Journal of Mathematical Sciences: Advances and Applications
Volume 5, Number 1, 2010, Pages 39-83

ASYMPTOTIC BEHAVIOR OF RELAXED DIRICHLET PROBLEMS RELATED TO *p*-HOMOGENEOUS STRONGLY LOCAL FORMS

MARCO BIROLI and SILVANA MARCHI

Dipartimento di Matematica "F. Brioschi"
Politecnico di Milano
Piazza L. Da Vinci 32, Milano
Italy
e-mail: marco.biroli@polimi.it

Dipartimento di Matematica
Università di Parma
Viale Usberti
53/A, 43100 Parma
Italy

Abstract

We study the asymptotic behavior of the solutions to a relaxed Dirichlet problem associated with p-homogeneous strongly local forms, $p > 1$, having a local L^1- density and to measures ζ_n, which do not charge sets of zero capacity. We prove that there exists a subsequence of ζ_n that γ- converges to a measure ζ of the same type, and we also prove the convergence of the relative solutions in $D^r[\Omega]$, $1 < r < p$.

2010 Mathematics Subject Classification: 31C45, 35B27, 31C05, 35J60, 35H20.

Keywords and phrases: nonlinear Dirichlet forms, relaxed Dirichlet problems, γ- convergence.

The authors have been supported by the MIUR Research Project no. 2007 WECYEA.

Received March 9, 2010

1. Introduction

The present paper is focused on the asymptotic behavior of the solutions to a relaxed Dirichlet problem associated with p-homogeneous strongly local forms of Riemannian type. In [1], it has been proved that the class of relaxed Dirichlet problems associated with p-homogeneous strongly local forms of Riemannian type in a r.c. open set is compact with respect to the γ-convergence. Here, under additional assumptions, we prove the compactness of the class of relaxed Dirichlet problems associated with p-homogeneous strongly local forms of Riemannian type in a r.c. open set Ω with respect to the convergence in $D^r[\Omega]$, $1 < r < p$, (see the end of the section for the definition), and we give a sort of corrector for our problem. The lines of proof are a refinement, adapted to our framework, of the ones in [9]. We recall that the case of bilinear Dirichlet forms of Riemannian type has been studied in [8] under slight stronger assumptions. Our framework applies to the subelliptic p-Laplacian eventually with a weight in the intrinsic A_p Muckenhoupt's class, or to the metric p-Laplacian, in the case, where the related norm (in the domain) defines a uniformly convex space. In the following of this section, we recall the basic definitions and properties relative to our framework.

We consider a locally compact connected complete separable Hausdorff space X with a metrizable topology and a positive Radon measure m on X such that $\operatorname{supp}[m] = X$. We observe that every bounded set in X is r.c.. We consider a strongly local p-homogeneous Dirichlet form, $p > 1$, $\int_X \mu(u, v)(dx)$ as defined in [5] $\left(\alpha(u) = \frac{1}{p}\mu(u, u)\right)$. We denote by $D_0 \subset L^p(X, m)$, the domain of the form endowed with the natural norm. The strong locality property allows us to define the domain of the form with respect to an open set O, denoted by $D_0[O]$, and the local domain of the form with respect to an open set O, denoted by $D_{loc}[O]$. Associated with the form a capacity $cap_p(E, O)$ can be defined and it can be proved that every function in D_0 is quasi-continuous and is defined quasi-everywhere [5].

We just list the main properties of strongly local p-homogeneous Dirichlet forms and we refer for the proofs to [5]:

(a) $\mu(u, v)$ is homogeneous of degree $p-1$ in u and linear in v; we also have $\mu(u, u) = p\alpha(u)$.

(b) Chain rule: if $u, v \in D_0$ and $g \in C^1(\mathbf{R})$ with $g(0) = 0$ and g' bounded on $\mathbf{R}$, then $g(u)$, $g(v)$ belong to D_0, and

$$\mu(g(u), v) = |g'(u)|^{p-2} g'(u)\,\mu(u, v). \tag{1.1}$$

Moreover,

$$\mu(u, g(v)) = g'(v)\,\mu(u, v). \tag{1.2}$$

Then

$$\alpha(g(u)) = |g'(u)|^{p}\,\alpha(u). \tag{1.3}$$

The assumption on the boundness of g' can be replaced by the assumption $u, v \in L^\infty(X, m)$.

(c) Truncation property: for every $u, v \in D_0$

$$\mu(u^+, v) = \mathbf{1}_{\{u>0\}}\mu(u, v), \tag{1.4}$$

$$\mu(u, v^+) = \mathbf{1}_{\{v>0\}}\mu(u, v), \tag{1.5}$$

where the above relations make sense, since u and v are defined quasi-everywhere.

(d) Leibniz rule with respect to the second argument:

$$\mu(u, vw) = v\mu(u, w) + w\mu(u, v), \tag{1.6}$$

where $u \in D_0$, $v, w \in D_0 \cap L^\infty(X, m)$.

(e) Leibniz inequality: let $u, v \in D_0 \cap L^\infty(X, m)$, then $uv \in D_0 \cap L^\infty(X, m)$, and

$$\alpha(uv) \leq C(|u|^p \alpha(v) + |v|^p \alpha(u)),$$

where $u \in D_0$, $v, w \in D_0 \cap L^\infty(X, m)$.

(f) For any $f \in L^{p'}(X, \alpha(u))$ and $g \in L^p(X, \alpha(v))$ with $1/p + 1/p =$ 1, fg is integrable with respect to $|\mu(u, v)|$ and $\forall a \in \mathbf{R}^+$

$$|fg||\mu(u, v)|(dx) \leq 2^{p-1}a^{-p}|f|^{p'}\alpha(u)(dx) + 2^{p-1}a^{p(p-1)}|g|^p\alpha(v)(dx). \tag{1.7}$$

Assume that we are given a distance d on X, such that $\alpha(d) \leq m$ in the sense of the measures, and

(i) The metric topology induced by d is equivalent to the original topology of X, and we also assume for sake of simplicity that $\sup_{y\in X} d(x_0, y) = +\infty$ (we can replace this last assumption by: let Ω be the r.c. open set in consideration, there exists a point in Ω^c with positive distance from Ω).

(ii) Denoting by $B(x, r)$, the ball of center x and radius r (for the distance d), for every fixed compact set K, there exist positive constants c_0 and r_0 such that

$$m(B(x, r)) \leq c_0 m(B(x, s))\left(\frac{r}{s}\right)^\nu \quad \forall x \in K \quad \text{and} \quad 0 < s < r < r_0. \tag{1.8}$$

We assume without loss of generality, $p < \nu$.

From the properties of d, it follows that for any $x \in X$, there exists a function $\phi(\cdot) = \phi(d(x, .))$ such that $\phi \in D_0[B(x, 2r)]$, $0 \leq \phi \leq 1$, $\phi = 1$ on $B(x, r)$ and $\alpha(\phi) \leq \frac{2}{r^p} m$, [6].

We also assume that the following scaled *Poincaré inequality* holds: for every fixed compact set K, there exist positive constants c_1, r_1, and $k \geq 1$ such that for every $x \in K$ and every $0 < r < r_1$

$$\int_{B(x,r)} |u - \overline{u}_{x,r}|^p m(dx) \leq c_1 r^p \int_{B(x,kr)} \mu(u, u)(dx), \tag{1.9}$$

for every $u \in D_{loc}[B(x, kr)]$, where $\overline{u}_{x,r} = \frac{1}{m(B(x,r))} \int_{B(x,r)} um(dx)$.

A strongly local p-homogeneous Dirichlet form, such that the above assumptions hold, is called a *Riemannian Dirichlet form*.

From (1.9), we can easily deduce by standard methods that for every fixed r.c. set Ω,

$$\int_\Omega |u|^p m(dx) \leq c_2(\Omega) \int_\Omega \alpha(u)(dx),$$

for every $u \in D_0[\Omega]$, where c_2 depends only on Ω; then $\int_\Omega \alpha(u)(dx)$ is an equivalent norm on $D_0[\Omega]$. Moreover, the embedding of $D_0[\Omega]$ in $L^p(\Omega, m)$ is compact. The following technical lemma will be utilized in Section 7.

Proposition 1.1. *For every p-quasi-open set U in the open set Ω, there exists an increasing sequence of functions $v_n \in D_0(\Omega)$, which converges to $\mathbf{1}_U$ q.e. in Ω.*

Proof. Let U be quasi-open in Ω. Then, there exists a sequence $U_k \subset \Omega$ with $cap_p(U_k, \Omega) \leq \frac{1}{k}$ such that the sets $A_k = U \cup U_k$ are open. We can assume without loss of generality that the sequence U_k is decreasing.

Therefore, for every k, there exists an increasing sequence of non-negative functions $\phi_h^k \in L^\infty(\Omega) \cap D_0[\Omega]$ with $\alpha(\phi_h^k) \leq M_h^k$, converging to $\mathbf{1}_{A_k}$ pointwise q.e. in Ω.

Since for every k, we have $cap_p(U_k, \Omega) \leq \frac{1}{k}$, there exists $u_k \in D_0[\Omega]$ such that q.e. $u_k = 1$ in U_k, $0 \leq u_k \leq 1$ q.e. and

$\int_\Omega \alpha(u_k)(dx) \leq \frac{1}{k}$ (it is enough to choose u_k as the potential of U_k in Ω). This implies that a subsequence of u_k converges to 0 q.e.. Moreover, as $\phi_h^k \leq \mathbf{1}_{A_k}$, we have $(\phi_h^k - u_k)^+ \leq \mathbf{1}_U$ q.e. in Ω. Let us define

$$v_h = \max_{1 \leq k \leq h}(\phi_h^k - u_k)^+, \; \psi = \sup_h v_h.$$

Then $v_h \in D_0[\Omega]$, $v_h \geq 0$ q.e. in Ω, moreover, the sequence v_h is increasing and $\psi \leq \mathbf{1}_U$ q.e. in Ω.

On the other hand, for every $h \geq k$, we have $v_h \geq (\phi_h^k - u_k)$. As $U \subset A_k$, we obtain $\psi \geq (1 - u_k)$ q.e. in U. Taking the limit along a suitable subsequence, we obtain $\psi \geq 1$ q.e. in U. This shows $\psi = \mathbf{1}_U$, which concludes the proof.

2. The Space of Measures $\mathcal{M}_0^p(\Omega)$ and the Operator

2.1. The measures

We denote by $\mathcal{M}_0^p(\Omega)$, the set of all non-negative Borel measures ζ such that

(i) $\zeta(B) = 0$ for every Borel set $B \subset \Omega$ with $cap_p(B, \Omega) = 0$.

(ii) $\zeta(B) = \inf\{\zeta(U), U$ quasi-open, $B \subset U\}$.

Property (ii) is a weak regularity property of the measure ζ. Since any quasi-open set differs from a Borel set by a set of capacity zero, then $\zeta(U)$ is well defined when U is quasi-open and ζ satisfies (i), so condition (ii) makes sense. The condition (ii) will be essential in the proof of the uniqueness of the γ^μ-limit. Finally, we observe that every non-negative Radon measure on Ω is in the class $\mathcal{M}_0^p(\Omega)$.

If ζ is a non-negative Borel measure, then $L^r(\Omega, \zeta)$, $1 \leq r \leq +\infty$, will denote the usual Lebesgue space with respect to the measure ζ.

If $\zeta \in \mathcal{M}_0^p(\Omega)$, then the space $D_0[\Omega] \cap L^p(\Omega, \zeta)$ is well defined because the functions in $D_0[\Omega]$ are defined q.e. [5], and then ζ- almost everywhere in Ω. Moreover, the space $D_0[\Omega] \cap L^p(\Omega, \zeta)$ is a Banach space for the norm

$$\| u \|^p_{D_0[\Omega] \cap L^p(\Omega, \zeta)} = \| u \|^p_{D_0[\Omega]} + \| u \|^p_{L^p(\Omega, \zeta)}.$$

A non-negative Borel measure, which is finite on compact sets of Ω is a non-negative Radon measure on Ω. We say that a Radon measure σ belongs to $D^{-1}[\Omega]$, (where $D^{-1}[\Omega] = (D_0[\Omega])'$) if there exists $f \in D^{-1}[\Omega]$ such that

$$< f, \varphi >= \int_\Omega \varphi d\sigma, \tag{2.1}$$

for every $\varphi \in D_0[\Omega] \cap C_0(\Omega)$, where $< \cdot, \cdot >$ denotes the pairing between $D^{-1}[\Omega]$ and $D_0[\Omega]$. We identify σ and f. We observe that for every non-negative $f \in D^{-1}[\Omega]$, there exists a non-negative Radon measure σ such that (2.1) holds. The proof is analogous to the one for distributions in euclidean spaces and is founded on the density of $D_0[\Omega] \cap C_0(\Omega)$ both in $D_0[\Omega]$ and in $C_0(\Omega)$ for the uniform convergence. Moreover, every non-negative Radon measure in $D^{-1}[\Omega]$ belongs to $\mathcal{M}_0^p(\Omega)$.

2.2. Properties of the energy density of the form

We will assume that, for any $u \in D_0$, the Radon measure $\alpha(u)$ has a density in $L^1(X, m)$, denoted again by $\alpha(u)(x)$. By (1.7), we obtain that for any $u, v \in D_0$, the Radon measure $\mu(u, v)$ has a density in $L^1(X, m)$, denoted again by $\mu(u, v)(x)$.

We also assume that there exist some constants $C_0, C_1 > 0$ such that for any $u_1, u_2, v \in D_0$

$$\mu(u_1, u_1 - u_2) - \mu(u_2, u_1 - u_2) \geq C_0\alpha(u_1 - u_2), \tag{2.2}$$

$$|\mu(u_1, v) - \mu(u_2, v)| \tag{2.3}$$

$$\leq C_1\left(\alpha(u_1)^{\frac{1}{p}} + \alpha(u_2)^{\frac{1}{p}}\right)^{p-2} \cdot \alpha(u_1 - u_2)^{\frac{1}{p}}\alpha(v)^{\frac{1}{p}},$$

a.e., if $p \geq 2$, and

$$\mu(u_1, u_1 - u_2) - \mu(u_2, u_1 - u_2) \tag{2.4}$$

$$\geq C_0\left(\alpha(u_1)^{\frac{1}{p}} + \alpha(u_2)^{\frac{1}{p}}\right)^{p-2} \cdot \alpha(u_1 - u_2)^{\frac{2}{p}},$$

$$|\mu(u_1, v) - \mu(u_2, v)| \leq C_1\alpha(u_1 - u_2)^{\frac{p-1}{p}}\alpha(v)^{\frac{1}{p}}, \tag{2.5}$$

a.e., if $1 < p < 2$. We also assume

$$\| u_1|^{p-2}u_1\mu(u_2, v) - \mu(u_1u_2, v)| \leq C|u_2|^{p-1}\alpha(u_1)^{\frac{p-1}{p}}\alpha(v)^{\frac{1}{p}}, \tag{2.6}$$

for any $u_1, u_2, v \in D_0$, $u_1, u_2 \in L^\infty(X, m)$.

The above conditions hold in the case where $\alpha(u) = \sum_{i=1}^m |L_i(u)|^p$, where $L_i : D_0 \to L^p(X, m)$ are linear bounded continuous operator, then in a framework similar to the one used in [1] in the bilinear case. In particular, our results can be applied to the case of the weighted subelliptic p-Laplacian, where the weight is in the corresponding intrinsic A_p Muckenhoupt's class (see [4] for the case without weight). Finally, the above assumptions hold for the p-Laplacian in finite dimensional metric structures of Cheeger type.

3. Relaxed Dirichlet Problems

Let Ω be a r.c. open set in X, $\zeta \in \mathcal{M}_0^p(\Omega)$, $f \in D^{-1}[\Omega]$, $\psi \in D[\Omega] \cap L^p(\Omega, \zeta)$, where $D[\Omega] = \{v \in D_{loc}[\Omega]; \int_\Omega \alpha(v)m(dx) < +\infty\}$. We also denote by $D^r[\Omega]$, $1 < r < p$, the closure of $D[\Omega]$ for the convergence defined as u_n converges to u in $D^r[\Omega]$, if u_n converges to u in $L^r(\Omega, m)$ and $\int_\omega \alpha(u_n - u)^{\frac{1}{p}}$ converges to 0 in $L^r(\Omega, m)$. We consider the following relaxed Dirichlet problem

$$\int_\Omega \mu(u, v)m(dx) + \int_\Omega | u |^{p-2} uv\zeta(dx) =< f, v > \tag{3.1}$$

$u \in D(\Omega) \cap L^p(\Omega, \zeta)$, $(u - \psi) \in D_0(\Omega)$, for every $v \in D_0(\Omega) \cap L^p(\Omega, \zeta)$. The problem (3.1) has a unique solution (see Theorem 4.1). We are interested in particular to the case $\psi = 0$, i.e., $u \in D_0[\Omega]$, in this case we refer to this problem as (3.1_0). In this paper we study the asymptotic behavior of relaxed Dirichlet problems (3.1_0) related to a sequence of measures $\zeta_n \in \mathcal{M}_0^p(\Omega)$.

Let ζ_n be a sequence in $\mathcal{M}_0^p(\Omega)$ and $\zeta \in \mathcal{M}_0^p(\Omega)$. Let $f \in D^{-1}[\Omega]$. Let u, u_n be the solutions of the problems

$$\int_\Omega \mu(u, v)m(dx) + \int_\Omega | u |^{p-2} uv\zeta(dx) =< f, v > \tag{3.2}$$

$u \in D_0[\Omega] \cap L^p(\Omega, \zeta_n)$, $\forall v \in D_0[\Omega] \cap L^p(\Omega, \zeta_n)$.

$$\int_\Omega \mu(u_n, v)m(dx) + \int_\Omega | u_n |^{p-2} u_n v\zeta_n(dx) =< f, v > \tag{3.2_n}$$

$u_n \in D_0[\Omega] \cap L^p(\Omega, \zeta)$, $\forall v \in D_0[\Omega] \cap L^p(\Omega, \zeta)$. Let w, w_n be respectively, the solutions of (3.2), (3.2_n) with $f = 1$. In Theorem 7.3, we prove that the following two assertions are equivalent.

(a) For every $f \in D^{-1}(\Omega) u_n$ converges to u weakly in $D_0(\Omega)$ (We say in this case that $\zeta_n \gamma^\mu$- converges to ζ in $\mathcal{M}_0^p(\Omega)$).

(b) w_n converges to w weakly in $D_0(\Omega)$.

We also prove that, if (a) holds, then the sequence u_n converges to u in $D^r[\Omega]$, for all $1 < r < p$. We observe that in Theorem 4.15, we also give a sort of correctors in $D_0[\Omega]$ for our problems. Theorem 7.5 is consequence of some compactness results, which are interesting in themselves proved in Section 7. In particular, we prove the compactness of the set of the solutions w of (3.2) with $f = 1$, when $\zeta \in \mathcal{M}_0^p(\Omega)$.

Theorems 7.3 and 7.5 are proved in Section 7. The previous sections contain many auxiliary results relative to the solutions of (3.2) and (3.2_n). In particular, in Section 5, we prove some estimates for the solutions of (3.2) and establish some comparison principles. The asymptotic behavior of certain sequences defined by the solutions of (3.2) and (3.2_n) are considered in Sections 5 and 6. Section 7 is devoted to prove the compactness results. The case of the Dirichlet problems in perforated domains is of particular interest. For every open set $U \subset \Omega$ and every Borel set $B \subset \Omega$, we define the non-negative Borel measure ζ_U as follows:

(j) $\zeta_U(B) = 0$, if $cap_p(B \cap U^c, \Omega) = 0$;

(jj) $\zeta_U(B) = +\infty$, otherwise.

Let Ω_n be an arbitrary sequence of open subset with closure contained in Ω. Let $f \in D^{-1}[\Omega]$ and denote by u_n the solutions to the problem

$$\int_{\Omega_n} \mu(u_n, v) m(dx) = < f, v >_{\Omega_n},$$

$u_n \in D_0(\Omega_n)$, for every $v \in D_0[\Omega_n]$ extended by 0 to Ω_n. Let us observe that the above equation is equivalent to the relaxed Dirichlet problem associated to the sequence of measures $\zeta_n = \zeta_{\Omega_n}$. From Theorem 7.5, we have that there exists a subsequence of Ω_n, still denoted by Ω_n, and a measure $\zeta \in \mathcal{M}_0^p(\Omega)$ such that for every $f \in D^{-1}[\Omega]$, the functions u_n extended by 0 to Ω, converges weakly in $D_0[\Omega]$ to the solution of the relaxed Dirichlet problem (3.2) relative to f and to the measure ζ.

4. Preliminaries Results

4.1. Estimates for the solutions of the relaxed problems

Proposition 4.1. *Let* $\zeta \in \mathcal{M}_0^p(\Omega)$, $\psi \in D[\Omega] \cap L^p(\Omega, \zeta)$. *The problem* (3.1) *has a unique solution. Moreover, the solution satisfies the estimate*

$$\int_\Omega \alpha(u)(dx) + \int_\Omega |u|^p \zeta(dx) \tag{4.1}$$

$$\leq \left(\| f \|^q_{D^{-1}(\Omega)} + \int_\Omega \alpha(\psi) m(dx) + \int_\Omega |\psi|^p \zeta(dx) \right),$$

where C is a structural constant.

Proof. Let $\int_\Omega \sigma(\cdot, \cdot)(dx)$ be the form defined as

$$\int_\Omega \sigma(z, v)(dx) = \int_\Omega \mu(z + \psi, v) m(dx) + \int_\Omega |z + \psi|^{p-2} (z + \psi) v \zeta(dx),$$

where $z, v \in D_0[\Omega] \cap L^p(\Omega, \zeta)$. From (2.2), ..., (2.5), the problem

$$\int_\Omega \sigma(z, v)(dx) = < f, v >_{D'[\Omega], D[\Omega]},$$

$z \in D_0[\Omega] \cap L^p(\Omega, \zeta)$, $\forall v \in D_0[\Omega] \cap L^p(\Omega, \zeta)$, admits a unique solution z. Then, $u = z + \psi$ is the solution of the problem (3.1). Let us take $v = (u - \psi)$ as test function in (3.1); we obtain

$$\int_\Omega \mu(u, u - \psi) m(dx) + \int_\Omega |u|^{p-2} u(u - \psi) \zeta(dx) = < f, u - \psi > .$$

Then, using the Young's inequality, we obtain (4.1).

The "uniform" continuous dependence of *f* on the solutions of (3.1) is given by the following theorem.

Proposition 4.2. *Let* $\zeta \in \mathcal{M}_0^p(\Omega)$, $u_1, u_2 \in D[\Omega] \cap L_\zeta^p(\Omega)$. *Let* $\varphi \in D[\Omega] \cap L^\infty(\Omega, m)$, $\varphi \geq 0$ *q.e. in* Ω. *If* $2 \leq p < \infty$,

$$C\int_\Omega \varphi\alpha(u_1 - u_2)m(dx) + 2^{2-p}\int_\Omega |u_1 - u_2|^p \varphi\zeta(dx)$$

$$\leq \int_\Omega \varphi(\mu(u_1, u_1 - u_2) - \mu(u_2, u_1 - u_2))m(dx)$$

$$+ \int_\Omega (|u_1|^{p-2}u_1 - |u_2|^{p-2}u_2)(u_1 - u_2)\varphi\zeta(dx).$$

If $1 < p < 2$,

$$C\left(\int_\Omega \varphi\alpha(u_1 - u_2)m(dx)\right)^{2/p}$$

$$\leq K_1(u_1, u_2, \varphi)\int_\Omega \varphi(\mu(u_1, u_1 - u_2) - \mu(u_2, u_1 - u_2))m(dx),$$

$$C\left(\int_\Omega |u_1 - u_2|^p \varphi\zeta(dx)\right)^{2/p}$$

$$\leq K_2(u_1, u_2, \varphi)\int_\Omega (|u_1|^{p-2}u_1 - |u_2|^{p-2}u_2)(u_1 - u_2)\varphi\zeta(dx),$$

where $K_1(u_1, u_2, \varphi) = 2(\int_\Omega \varphi\alpha(u_1)m(dx) + \int_\Omega \varphi\alpha(u_2)m(dx))^{\frac{2-p}{p}}$, $K_2(u_1, u_2, \varphi) = 2(\int_\Omega |u_1|^p \varphi\zeta(dx) + \int_\Omega |u_2|^p \varphi\zeta(dx))^{\frac{2-p}{p}}$.

Proof. The proof is the same of [4] and is founded on (2.2) ,..., (2.5).

Proposition 4.3. *Let* $\zeta \in \mathcal{M}_0^p(\Omega)$; *let* $f_1, f_2 \in D^{-1}[\Omega]$, *and let* u_1, u_2 *be the solutions of* (3.1) *corresponding to* f_1 *and* f_2, *respectively. If* $p \geq 2$, *then*

$$\|u_1 - u_2\|_{D_0[\Omega]}^p + \|u_1 - u_2\|_{L^p(\Omega, \zeta)}^p \leq C\|f_1 - f_2\|_{D^{-1}[\Omega]}^q. \tag{4.2}$$

If $1 < p < 2$, *then*

$$\|u_1 - u_2\|^2_{D_0[\Omega]} + \|u_1 - u_2\|^2_{L^p(\Omega, \zeta)} \leq C\, \Gamma(f_1, f_2, \psi) \|f_1 - f_2\|^2_{D^{-1}[\Omega]}, \quad (4.3)$$

where C *is a structural constant, and*

$$\Gamma(f_1, f_2, \psi) = \left(\|f_1\|^q_{D^{-1}(\Omega)} + \|f_2\|^q_{D^{-1}[\Omega]} + \int_\Omega \alpha(\psi) m(dx) + \int_\Omega |\psi|^p \zeta(dx) \right)^{\frac{2(2-p)}{p}}.$$

The proof follows as in [9] taking into account the assumptions (2.2), ..., (2.5).

4.2. Comparison principles

Proposition 4.4. *Let* $\zeta \in \mathcal{M}_0^p(\Omega)$; *let* $f \in D^{-1}[\Omega]$ *and let* u *be the solution of* (3.2). *If* $f \geq 0$ *in* Ω, *then* $u \geq 0$ *q.e. in* Ω.

Proof. The results follow by using $v = u \wedge 0$ as test function in (3.2).

Proposition 4.5. *Let* w_0 *be the solution of the problem*

$$\int_\Omega \mu(w_0, v) m(dx) = \int_\Omega v m(dx), \quad (4.4)$$

$w_0 \in D_0[\Omega]$, *for every* $v \in D_0[\Omega]$. *Then* $w_0 > 0$ *q.e. in* Ω.

Proof. The function w_0 is a non-negative superharmonic in Ω for the form μ, that is, $w_0 \geq 0$ and $\int_\Omega \mu(w_0, v) m(dx) \geq 0$, for every $v \in D_0[\Omega]$, $v \geq 0$. Then $(w_0 + \epsilon)$, $\epsilon > 0$, satisfies an A_2 Muckenhoupt's condition in every ball B such that $2B \subset \Omega$ with a constant independent of ϵ [6]. The result follows from [6], since $v_0 = (w_0 + \epsilon)^{-1}$ is non-negative and subharmonic in Ω for the form μ.

Proposition 4.6. *Let* $\zeta_1, \zeta_2 \in \mathcal{M}_0^p(\Omega)$; *let* $f_1, f_2 \in D^{-1}[\Omega]$, *and let* u_1, u_2 *be the respective solutions of* (3.2). *If* $0 \leq f_2$ *and* $\zeta_2 \leq \zeta_1$ *in* Ω, *then* $u_1 \leq u_2$ *q.e. in* Ω.

Proof. By Proposition 4.4, we have $u_2 \geq 0$ q.e. in Ω. Let $v = (u_1 - u_2)^+$. Since $0 \leq v \leq u_1^+$ and $\zeta_2 \leq \zeta_1$, we have $v \in L^p(\Omega, \zeta_1) \subset L^p(\Omega, \zeta_2)$. Then we can use v as test function in both the relaxed Dirichlet problems, and we obtain $\alpha(v) = 0$, so $u_1 \leq u_2$ q.e. in Ω.

Proposition 4.7. *Let* $\zeta_1, \zeta_2 \in \mathcal{M}_0^p(\Omega)$ *and* $f_1, f_2 \in D^{-1}[\Omega]$, *and let* u_1, u_2 *be the respective solutions of* (3.2). *If* $|f_1| \leq f_2$ *and* $\zeta_2 \leq \zeta_1$ *in* Ω, *then* $|u_1| \leq u_2$ *q.e. in* Ω.

Proof. By Proposition 4.6, we have $u_1 \leq u_2$ q.e. in Ω. We observe that the function $-u_1$ is the solution of (3.2) corresponding to $-f_1$ and ζ_1. So by Proposition 4.6, we also have $-u_1 \leq u_2$ q.e. in Ω.

Remark 4.8. Let $\zeta \in \mathcal{M}_0^p(\Omega)$ and let u_n and w_n be the solutions of the problem (3.2_n) relative to, $f \in L^\infty(\Omega)$ and to $f = 1$. From the Proposition 4.1, the sequences u_n and w_n are bounded in $D_0[\Omega]$. Then, there are subsequences still denoted by u_n and w_n, and two functions $u, w \in D_0[\Omega]$ such that u_n and w_n converge weakly in $D_0[\Omega]$ and a.e. in Ω to u and w. Let $C = \| f \|_{L^\infty(\Omega, m)}^{1/(p-1)}$. From ($3.1_0$), we have

$$\int_\Omega \mu\left(\frac{u_n}{C}, v\right) m(dx) + \int_\Omega \left| \frac{u_n}{C} \right|^{p-2} \frac{u_n}{C} v \zeta_n(dx) = \int_\Omega \frac{f}{\|f\|_{L^\infty(\Omega)}} v m(dx),$$

for every $v \in D_0[\Omega] \cap L^p(\Omega, \zeta_n)$. Proposition 4.6 gives $\frac{u_n}{C} \leq w_n$ q.e. in Ω. Let w_0 be the solution of (4.4). In virtue of the Proposition 4.6, we have $w_n \leq w_0$ q.e. in Ω. Then $|u_n| \leq C w_n \leq C w_0$ q.e. in Ω. Hence $|u| \leq Cw \leq Cw_0$ a.e. in Ω. As $w_0 \in L^\infty(\Omega, m)$, the sequences u_n and w_n are bounded in $L^\infty(\Omega, m)$.

4.3. Estimates involving auxiliary Radon measures

Proposition 4.9. *Let* $\zeta \in \mathcal{M}_0^p(\Omega)$; *let* $f \in L^q(\Omega, m)$, $q = \dfrac{p}{p-1}$, *and let* u *be the solution of* (3.1) *for some* $\psi \in D[\Omega] \cap L^p(\Omega, \zeta)$. *Let* $\lambda, \lambda_1, \lambda_2$ *be elements of* $D^{-1}[\Omega]$ *defined by* $\int_\Omega \mu(u, v)m(dx) = \int_\Omega fvm(dx) + < \lambda, v >$, $\int_\Omega \mu(u^+, v)m(dx) = \int_\Omega f^+vm(dx) + < \lambda_1, v >$, $\int_\Omega \mu(u^-, v)m(dx) = \int_\Omega f^-vm(dx) + < \lambda_2, v >$, $\forall v \in D_0[\Omega]$. *Then* $\lambda, \lambda_1, \lambda_2$ *are Radon measures,* $\lambda_1, \lambda_2 \geq 0$, $\lambda = \lambda_1 - \lambda_2$, $|\lambda| \leq \lambda_1 + \lambda_2$. *Moreover, for every compact set* $K \subset \Omega$, *we have*

$$|\lambda(K)| \leq C\ cap_p(K, \Omega)^{1/p}\left[\| u \|_{D_0(\Omega)}^{p-1} + \| f \|_{L^q(\Omega, m)}\right]. \tag{4.5}$$

Proof. Let $v \in D_0[\Omega]$, $v \geq 0$ q.e. in Ω and let $v_n = (\frac{v}{n}) \wedge u^+$. Then $v_n \geq 0$ q.e. in Ω, $v_n \in D_0[\Omega] \cap L^p(\Omega, \zeta)$. As $|u|^{p-2}uv_n \geq 0$ q.e. in Ω, and $fv_n \leq f^+v_n$ a.e. in Ω, taking v_n as test function in (3.1), we obtain $(v_n = 0$ if $u \leq 0)$

$$\int_\Omega \mu(u^+, v_n)m(dx) \leq \int_\Omega f^+v_nm(dx) \leq \frac{1}{n}\int_\Omega f^+vm(dx),$$

where we use the truncation rule. Since by the truncation rule $\mu(u^+, v_n) = \frac{1}{n}\mu(u^+, v)$ in $\{v < nu^+\}$ and $\mu(u^+, v_n) = \mu(u^+, u^+)$ in $\{v \geq nu^+\}$, we obtain

$$\frac{1}{n}\int_{\{v<nu^+\}} \mu(u^+, v)m(dx) + \frac{1}{n}\int_{\{v\geq nu^+\}} \alpha(u^+)m(dx) \leq \frac{1}{n}\int_\Omega f^+vm(dx).$$

Taking the limit as $n \to +\infty$, we obtain

$$\int_\Omega \mu(u^+, v)m(dx) = \int_{\{u^+>0\}} \mu(u, v)m(dx) \leq \int_\Omega f^+vm(dx),$$

for every $v \in D_0[\Omega]$, $v \geq 0$ q.e. in Ω. This implies $< \lambda_1, v > \geq 0$, so, since $\lambda_1 \in D'[\Omega]$, λ_1 is a non-negative Radon measure.

In a similar way, we also deduce that λ_2 is a non-negative Radon measure, hence $\lambda = \lambda_1 - \lambda_2$ is also a Radon measure and $|\lambda| \le \lambda_1 + \lambda_2$.

We prove (4.5) in the case $1 < p < 2$; the proof in the case $p \ge 2$ is similar. To prove (4.5) for every $\epsilon > 0$, we fix a function $z \in D_0[\Omega]$ such that $z \ge 0$ q.e. in Ω, $z \ge 1$ q.e. in a neighborhood of K and $\| z \|^p_{D_0[\Omega]} \le cap_p(K, \Omega) + \epsilon$.

$$
\begin{aligned}
|\lambda(K)| &= |\lambda_1(K) - \lambda_2(K)| \\
&= |\int_\Omega \mu(u^+, z)m(dx) - \int_\Omega \mu(u^-, z)m(dx) \\
&\quad + \int_\Omega f^+ zm(dx) - \int_\Omega f^- zm(dx)| \\
&\le C\int_\Omega \alpha(z)^{\frac{1}{p}} \alpha(u)^{\frac{p-1}{p}} m(dx) + C\| f \|_{D^{-1}[\Omega]} \| z \|_{D_0[\Omega]} \\
&\le C\| z \|_{D_0[\Omega]} \| u \|^{p-1}_{D_0[\Omega]} + C\| f \|_{D^{-1}[\Omega]} \| z \|_{D_0[\Omega]} \\
&\le C(cap_p(K, \Omega) + \epsilon)^{1/p}\Big[\| u \|^{p-1}_{D_0[\Omega]} + \| f \|_{L^q(\Omega, m)}\Big].
\end{aligned}
$$

Taking the limit as $\epsilon \to 0$, we obtain (4.5).

Remark 4.10. Under the assumptions of Proposition 4.9, if $f, \psi \ge 0$, then $u = u^+$ and $\lambda = \lambda_1$. Therefore, in this case, $\lambda \ge 0$. Hence, $\int_\Omega \mu(u, v) \le \int_\Omega fvm(dx)$ for every $v \ge 0$ in $D_0[\Omega]$.

Proposition 4.11. *Let* g_n *be a sequence in* $D^{-1}[\Omega]$, *let* λ_n *be a sequence of Radon measures and let* $u_n \in D[\Omega]$ *be such that*

$$
\int_\Omega \mu(u_n, v)m(dx) =< g_n, v >_{D'[\Omega], D_0[\Omega]} + \int_\Omega v\lambda_n(dx),
$$

for every $v \in D_0[\Omega] \cap C_0(\Omega)$. *Assume that* u_n *converges weakly in* $D[\Omega]$ *to some function* u, g_n *converges in* $D^{-1}[\Omega]$ *and* λ_n *is bounded in the space of Radon measures (i.e., for every compact set* $K \subset \Omega$, *there exists a constant* C_K *such that* $|\lambda_n(K)| \le C_K$). *Then, for* $1 < r < p$, u_n *converges to* u *in* $D^r[\Omega]$; *moreover,* $\int_\Omega \mu(u_n, v)m(dx)$ *converges to* $\int_\Omega \mu(u, v)m(dx)$ *for every* $v \in D_0[\Omega]$.

The proof of this result is given in the Appendix.

Proposition 4.12. *Let* g_n *be a sequence in* $D^{-1}[\Omega]$, *which converges to some* $g \in D^{-1}[\Omega]$, *let* ζ_n *be a sequence in* $\mathcal{M}_0^p(\Omega)$, *and let* ψ_n *be a sequence bounded in* $D[\Omega] \cap L^p(\Omega, m)$ *such that* $\int_\Omega |\psi_n|^p \zeta_n(dx) \le M$. *Assume that the solution* u_n *of* (3.1) *corresponding to* $\zeta = \zeta_n$, $f = g_n$, $\psi = \psi_n$ *converges weakly in* $D[\Omega]$ *to some function* u. *Then, for* $1 < r < p$, u_n *converges to* u *in* $D^r[\Omega]$; *moreover,* $\int_\Omega \mu(u_n, v)m(dx)$ *converges to* $\int_\Omega \mu(u, v)m(dx)$ *for every* $v \in D_0[\Omega]$.

Proof. Let $g \in L^q(\Omega, m)\,(q = \frac{p}{p-1})$; then from Propositions 4.10 and 4.11, the result follows. In the general case, the result is proved by an approximation of g by a function f in $L^q(\Omega, m)$ by using the Proposition 4.3.

Proposition 4.13. *Let* $\zeta_n \in \mathcal{M}_0^p(\Omega)$ *be a sequence of measures. Let* u_n *and* w_n *be the solutions of the problem* (3.2_n) *relative to* $f \in L^\infty(\Omega)$ *and* $f = 1$. *Assume that* u_n *and* w_n *converge weakly in* $D_0[\Omega]$ *to some functions* u *and* w. *For every* $\epsilon > 0$, *the functions* $\frac{uw_n}{w \vee \epsilon}$ *belong to* $D_0[\Omega] \cap L^p(\Omega, \zeta_n)$, *and one has*

$$\lim_{n\to+\infty}\left(\int_{U_\epsilon}\alpha\left(u_n - \frac{uw_n}{w\vee\epsilon}\right)m(dx) + \int_{U_\epsilon}|u_n - \frac{uw_n}{w\vee\epsilon}|^p\zeta_n(dx)\right) = 0, \quad (4.6)$$

where $U_\epsilon = \{w > \epsilon\}\cap\{|u| > \epsilon w\}$.

Proof. For $\epsilon > 0$ denote

$$u_n^\epsilon = \frac{uw_n}{w\vee\epsilon}, \qquad r_n^\epsilon = u_n - u_n^\epsilon.$$

First step. We observe that the functions u_n and w_n (u and w) are bounded in $L^\infty(\Omega, m)$ (Remark 4.10) and converge to u and w weakly in $D_0[\Omega]$ and strongly in $L^p(\Omega, m)$. The functions u_n^ϵ and r_n^ϵ are bounded in $L^\infty(\Omega)$ (as $f\in L^\infty(\Omega, m)$ (Remark 4.10) and converge to $\frac{uw}{w\vee\epsilon}$ and $u - \frac{uw}{w\vee\epsilon}$ weakly in $D_0[\Omega]$ and strongly in $L^p(\Omega, m)$. Moreover, from Proposition 4.12, $\alpha(u_n - u)^{\frac{1}{p}}(\alpha(u_n^\epsilon - \frac{uw}{w\vee\epsilon})^{\frac{1}{p}})$ converges to 0 in $L^r(\Omega)$, $1\le r < p$ as $n\to+\infty$.

We recall that $D_0[\Omega]\cap L^\infty(\Omega, m)\cap L^p(\Omega, \zeta)\subset L^\infty(\Omega, \zeta)$, for every $\zeta\in\mathcal{M}_0^p(\Omega)$. Moreover, we have $u_n\in L^p(\Omega, \zeta_n)$, then $u_n^\epsilon, r_n^\epsilon\in L^p(\Omega, \zeta_n)$. As $u - \frac{uw}{w\vee\epsilon} = 0$ a.e. in U_ϵ, we obtain that r_n^ϵ converges to 0 strongly in $L^p(U_\epsilon, m)$ as $n\to+\infty$.

Consider now a Lipschitz function Φ_ϵ defined by $\Phi_\epsilon(t) = 0$ for $t\le\epsilon$, $\Phi_\epsilon(t) = \frac{t}{\epsilon} - 1$ for $\epsilon\le t\le 2\epsilon$, $\Phi_\epsilon(t) = 1$ for $t\ge 2\epsilon$. We define $\phi = \Phi_\epsilon(w)\Phi_\epsilon(\frac{|u|}{w\vee\epsilon})$. We have $\phi\in D_0[\Omega]\cap L^\infty(\Omega, m)$, $0\le\phi\le 1$ q.e. in Ω, $\phi = 1$ in $U_{2\epsilon}$, $\phi = 0$ in $\Omega\setminus U_\epsilon$.

By the previous remarks by using the Leibniz inequality, the sequence $r_n^\epsilon\phi$ converges to 0 weakly in $D_0[\Omega]$ and strongly in $L^p(\Omega, m)$.

Second step. We define

$$E_n^\epsilon = \int_\Omega \phi(\mu(u_n, r_n^\epsilon) - \mu(u_n^\epsilon, r_n^\epsilon))m(dx)$$

$$+ \int_\Omega r_n^\epsilon \phi(|u_n|^{p-2}u_n - |u_n^\epsilon|^{p-2}u_n^\epsilon)\,\zeta_n(dx).$$

In this step, we prove that for ϵ fixed, we have

$$\lim_{n\to+\infty} E_n^\epsilon = 0.$$

We write E_n^ϵ as

$$E_n^\epsilon = \int_\Omega (\mu(u_n, \phi r_n^\epsilon) - \mu(u_n^\epsilon, \phi r_n^\epsilon))m(dx) \tag{4.7}$$

$$+ \int_\Omega r_n^\epsilon \phi(|u_n|^{p-2}u_n - |u_n^\epsilon|^{p-2}u_n^\epsilon)\,\zeta_n(dx)$$

$$- \int_{U_\epsilon} r_n^\epsilon(\mu(u_n, \phi) - \mu(u_n^\epsilon, \phi))m(dx)$$

$$= \int_\Omega \mu(u_n, \phi r_n^\epsilon)m(dx) + \int_\Omega r_n^\epsilon \phi |u_n|^{p-2}u_n \zeta_n(dx)$$

$$- \int_\Omega \left|\frac{u}{w\vee\epsilon}\right|^{p-2}\frac{u}{w\vee\epsilon}\,\mu(w_n, \phi r_n^\epsilon)m(dx) - \int_\Omega \phi r_n^\epsilon |u_n^\epsilon|^{p-2}u_n^\epsilon \zeta_n(dx)$$

$$+ \int_{U_\epsilon}\left(\left|\frac{u}{w\vee\epsilon}\right|^{p-2}\frac{u}{w\vee\epsilon}\,\mu(w_n, \phi r_n^\epsilon) - (u_n^\epsilon, \phi r_n^\epsilon)\right)m(dx)$$

$$- \int_{U_\epsilon} r_n^\epsilon(\mu(u_n, \phi) - \mu(u_n^\epsilon, \phi))m(dx).$$

We have $w \le \epsilon$ in $\Omega \setminus U_\epsilon$, then $\Phi_\epsilon(w) = 0$ and so $\phi = 0$ q.e. in $\Omega \setminus U_\epsilon$. Then the function $\left|\frac{u}{w\vee\epsilon}\right|^{p-2}\frac{u}{w\vee\epsilon}\phi \in D_0[\Omega] \cap L^\infty(\Omega, m)$. We have

$$- \int_\Omega \left|\frac{u}{w\vee\epsilon}\right|^{p-2}\frac{u}{w\vee\epsilon}\,\mu(w_n, \phi r_n^\epsilon)m(dx)$$

$$= -\int_\Omega \mu(w_n, |\frac{u}{w \vee \epsilon}|^{p-2}\frac{u}{w \vee \epsilon}\phi r_n^\epsilon)m(dx)$$

$$+ (p-1)\int_\Omega |\frac{u}{w \vee \epsilon}|^{p-2}\phi r_n^\epsilon \mu(w_n, \frac{u}{w \vee \epsilon})m(dx).$$

Then, taking as test function $v = |\frac{u}{w \vee \epsilon}|^{p-2}\frac{u}{w \vee \epsilon}\phi r_n^\epsilon$ in the equation defining w_n, we obtain ($\phi = 0$ and $\alpha(\phi) = 0$ in $\Omega \setminus U_\epsilon$)

$$-\int_\Omega \mu(w_n, |\frac{u}{w \vee \epsilon}|^{p-2}\frac{u}{w \vee \epsilon}\phi r_n^\epsilon)m(dx) - \int_\Omega r_n^\epsilon \phi |u_n^\epsilon|^{p-2}u_n^\epsilon \zeta_n(dx)$$

$$= -\int_{U_\epsilon} |\frac{u}{w \vee \epsilon}|^{p-2}\frac{u}{w \vee \epsilon}\phi r_n^\epsilon m(dx).$$

Taking $v = \phi r_n^\epsilon$ as test function in the equation defining u_n, we obtain (taking into account that $\phi = 0$ and $\alpha(\phi) = 0$ in $\Omega \setminus U_\epsilon$).

$$E_n^\epsilon = \int_{U_\epsilon} f\phi r_n^\epsilon m(dx) - \int_{U_\epsilon} |\frac{u}{w \vee \epsilon}|^{p-2}\frac{u}{w \vee \epsilon}\phi r_n^\epsilon m(dx)$$

$$+ (p-1)\int_{U_\epsilon} |\frac{u}{w \vee \epsilon}|^{p-2}\phi r_n^\epsilon \mu(w_n, \frac{u}{w \vee \epsilon})m(dx)$$

$$+ \int_{U_\epsilon}\left(|\frac{u}{w \vee \epsilon}|^{p-2}\frac{u}{w \vee \epsilon}\mu(w_n, \phi r_n^\epsilon) - \mu(u_n^\epsilon, \phi r_n^\epsilon)\right)m(dx)$$

$$- \int_{U_\epsilon} r_n^\epsilon(\mu(u_n, \phi) - \mu(u_n^\epsilon, \phi))m(dx)$$

$$= I_n^1 + I_n^2 + I_n^3 + I_n^4 - I_n^5.$$

Let us recall that since r_n^ϵ is bounded in $D_0[\Omega]$, it converges strongly to 0 in $L^p(U_\epsilon, m)$, then a.e. in U_ϵ. Moreover, u_n^ϵ and $\frac{u}{w \vee \epsilon}$ are bounded in $D_0[\Omega]$ and $\frac{u}{w \vee \epsilon}$ is bounded in $L^\infty(\Omega, m)$.

It follows that I_n^1, I_n^2, I_n^3, I_n^5 converge to 0. The Young's inequality gives the result about I_n^1 and I_n^2. Concerning I_n^3, we recall that $\mu(w_n, \frac{u}{w \vee \epsilon})$ converges in $L^1(\Omega, m)$ (see Remark A.2) and the result easily follows. Concerning I_n^5, the method of the proof is the same, since $\mu(u_n, \phi)$ and $\mu(u_n^\epsilon, \phi)$ converge in $L^1(\Omega, m)$ (see the Appendix). Concerning I_n^4, the result follows as in [9], taking into account (2.6).

Third step. If $p \geq 2$, the Theorem 4.2 gives

$$\int_\Omega \phi\alpha(r_n^\epsilon)m(dx) + 2^{2-p}\int_\Omega |r_n^\epsilon|^p \phi\zeta_n(dx) \leq E_n^\epsilon, \tag{4.8}$$

and the proposition follows by Step 2. If $1 < p < 2$, we observe that the sequences $\|u_n\|_{L^p(\Omega, \zeta_n)}$ and $\|w_n\|_{L^p(\Omega, \zeta_n)}$ are bounded by Theorem 4.1. Since u and $\frac{1}{w \vee \epsilon}$ belong to $D_0[\Omega] \cap L^\infty(\Omega, m)$, we conclude that $\|u_n^\epsilon\|_{L^p(\Omega, \zeta_n)}$ and $\|r_n^\epsilon\|_{L^p(\Omega, \zeta_n)}$ are bounded too. By Theorem 4.2, there exists a constant K such that

$$\int_\Omega \phi\alpha(r_n^\epsilon)m(dx) + 2^{2-p}\int_\Omega |r_n^\epsilon|^p \phi\zeta_n(dx) \leq (KE_n^\epsilon)^{p/2}. \tag{4.9}$$

Taking (4.8) and (4.9) into account, we obtain from the Step 2 that for every $p > 1$,

$$\lim_{n\to+\infty}\left(\int_{U_{2\epsilon}} \alpha(r_n^\epsilon)m(dx) + 2^{2-p}\int_{U_{2\epsilon}} |r_n^\epsilon|^p \zeta_n(dx)\right) = 0. \tag{4.10}$$

As $w \vee 2\epsilon = w \vee \epsilon$ q.e. in $U_{2\epsilon}$, we have $r_n^\epsilon = u_n - \frac{uw_n}{w \vee 2\epsilon}$ q.e. in $U_{2\epsilon}$. Therefore, (4.10) implies (4.6) with ϵ replaced by 2ϵ.

Proposition 4.14. *Let $f \in L^\infty(\Omega, m)$, let u_n, w_n, u, and w be as in Proposition* 4.13. *For every $\epsilon > 0$, define $V_\epsilon = \{w \leq \epsilon\}$. Then*

$$\lim_{\epsilon\to 0}\lim_{n\to+\infty}\left(\int_{V_\epsilon} \alpha(u_n)m(dx) + \int_{V_\epsilon} |u_n|^p \zeta_n(dx)\right) = 0. \tag{4.11}$$

Proof. For every $\epsilon > 0$, let Φ_ϵ be the Lipschitz function defined at the end of the first step of Proposition 4.12, and let $z^\epsilon \in D_0[\Omega] \cap L^\infty(\Omega, m)$ be the function defined by $z^\epsilon = 1 - \Phi_\epsilon(w)$. As $z^\epsilon \geq 1$ q.e. in Ω and $z^\epsilon = 1$ q.e. in V_ϵ by (3.2n), we have

$$\int_{V_\epsilon} \alpha(u_n)m(dx) + \int_{V_\epsilon} |u_n|^p \zeta_n(dx)$$

$$\leq \int_\Omega z^\epsilon \alpha(u_n)m(dx) + \int_{V_\epsilon} |u_n|^p z^\epsilon \zeta_n(dx)$$

$$= \int_\Omega \mu(u_n, u_n z^\epsilon)m(dx) + \int_\Omega |u_n|^p z^\epsilon \zeta_n(dx) - \int_\Omega u_n \mu(u_n, z^\epsilon)m(dx)$$

$$= \int_\Omega f u_n z^\epsilon m(dx) - \int_\Omega u_n \mu(u_n, z^\epsilon)m(dx).$$

Let us observe that u_n converges strongly to u in $L^p(\Omega, m)$, and then a.e. in Ω and it is bounded in $L^\infty(\Omega, m)$. Moreover, $\mu(u_n, z^\epsilon) \to \mu(u, z^\epsilon)$ in $L^1(\Omega, m)$ (see Remark A.2). Then $\int_\Omega u_n \mu(u_n, z^\epsilon)m(dx) \to \int_\Omega u\mu(u, z^\epsilon)m(dx)$. Finally, we obtain

$$\lim_{n\to+\infty}\left(\int_{V_\epsilon} \alpha(u_n)m(dx) + \int_{V_\epsilon} |u_n|^p \zeta_n(dx)\right) \tag{4.12}$$

$$\leq \int_\Omega f u z^\epsilon m(dx) - \int_\Omega u\mu(u, z^\epsilon)m(dx).$$

Let us observe that z^ϵ is bounded in $L^\infty(\Omega, m)$ and converges to the characteristic function of the set $\{u = 0\}$ as $\epsilon \to 0$. Then uz^ϵ converges to 0 strongly in $L^p(\Omega, m)$. Let us observe that $\operatorname{supp} z^\epsilon = \{0 < |w| < 2\epsilon\}$, then

$$\lim_{\epsilon\to 0} \int_\Omega | u |^p \alpha(z^\epsilon)m(dx) = 0.$$

Taking the limit $\epsilon \to 0$ in (4.12), we obtain (4.11).

Proposition 4.15. *Let* $f \in L^\infty(\Omega, m)$, *let* u_n, w_n, u, *and* w *be as in Proposition* 4.13. *For every* $\epsilon > 0$, *define* $W_\epsilon = \{w > \epsilon\} \cap \{|u| \le \epsilon w\}$. *Then*

$$\lim_{\epsilon\to 0} \lim_{n\to+\infty}\left(\int_{W_\epsilon} \alpha(u_n)m(dx) + \int_{W_\epsilon} |u_n|^p \zeta_n(dx)\right) = 0. \qquad (4.13)$$

Proof. For every $\epsilon > 0$, let Φ_ϵ be the Lipschitz function defined at the end of the first step of Proposition 4.13. As $\frac{u}{w \vee \epsilon} \in D_0[\Omega] \cap L^\infty(\Omega, m)$, the function $z^\epsilon = 1 - \Phi_\epsilon\left(\frac{|u|}{w \vee \epsilon}\right)$ belongs to $D[\Omega] \cap L^\infty(\Omega, m)$. Since $z^\epsilon \ge 0$ q.e. in Ω and $z^\epsilon = 1$ on W_ϵ, by the same computations as in Proposition 4.13, we obtain

$$\lim_{n\to+\infty}\left(\int_{W_\epsilon} \alpha(u_n)m(dx) + \int_{W_\epsilon} |u_n|^p \zeta_n(dx)\right) \qquad (4.14)$$

$$\le \int_\Omega fuz^\epsilon m(dx) - \int_\Omega u\mu(u, z^\epsilon)m(dx).$$

Let us observe that z^ϵ is bounded in $L^\infty(\Omega, m)$ and converges to the characteristic function of the set $\{u = 0\}$. Then uz^ϵ converges to 0 strongly in $L^p(\Omega, m)$. Moreover, $\operatorname{supp} z^\epsilon \subset \{0 < |u| < 2\epsilon(w \vee \epsilon)\}$. We can now end the proof by the same computations as in Proposition 4.14.

From Propositions 4.13, 4.14, and 4.15, it follows:

Theorem 4.15. *Let* $\zeta_n \in \mathcal{M}_0^p(\Omega)$ *be a sequence of measures. Let* u_n, w_n, u, *and* w *be as in Proposition* 4.13. *Assume that* u_n *and* w_n *converge weakly in* $D_0[\Omega]$ *to some functions* u *and* w. *We have*

$$\lim_{\epsilon\to 0} \lim_{n\to+\infty}\left(\int_\Omega \alpha\left(u_n - \frac{uw_n}{w \vee \epsilon}\right)m(dx) + \int_\Omega \left|u_n - \frac{uw_n}{w \vee \epsilon}\right|^p \zeta_n(dx)\right) = 0.$$

The function $\frac{uw_n}{w \vee \epsilon}$ defines a corrector in $D_0[\Omega]$ for our problem.

5. Asymptotic Behavior of Certain Sequences

Let $\zeta_n \in \mathcal{M}_0^p(\Omega)$ and $f \in L^\infty(\Omega, m)$. Assume that u_n and w_n are the solutions of the problem (3.2n) relative to f and $f = 1$, and that u_n and w_n converge weakly in $D_0[\Omega]$ to some functions u and w. In this section, we will study the behavior of the following sequences

$$\mu(u_n, w_n^\beta \varphi) - \mu(w_n, |\frac{u}{w \vee \epsilon}|^{p-2} \frac{u}{w \vee \epsilon} w_n^\beta \varphi), \tag{5.1}$$

$$\int_\Omega |u_n|^{p-2} u_n w_n^\beta \varphi \zeta_n(dx) - \int_\Omega |\frac{u}{w \vee \epsilon}|^{p-2} \frac{u}{w \vee \epsilon} w_n^{p-1+\beta} \varphi \zeta_n(dx), \tag{5.2}$$

where $\beta \geq (p-1) \vee 1$ and $\varphi \in D_0[\Omega] \cap L^\infty(\Omega, m)$. The estimates will be useful in the proofs in the following sections of the paper. For $1 < p < 2$, the function $|\frac{u}{w \vee \epsilon}|^{p-2} \frac{u}{w \vee \epsilon}$ does not belong to $D_0[\Omega]$, then the formula (5.1) and (5.2) are not correct. We introduce the locally Lipschitz function $\Psi_\epsilon(t)$ defined by:

$$\Psi_\epsilon(t) = |t|^{p-2} t \text{ if } |t| > \epsilon, \quad \Psi_\epsilon(t) = \epsilon^{p-2} t \text{ if } |t| \leq \epsilon, \tag{5.3}$$

and we replace in (5.1), (5.2) $|\frac{u}{w \vee \epsilon}|^{p-2} \frac{u}{w \vee \epsilon}$ by $\Psi_\epsilon(\frac{u}{w \vee \epsilon})$. We begin with an estimate in $U_\epsilon = \{w > \epsilon\} \cap \{|u| > \epsilon w\}$.

Lemma 5.1. *Let $\epsilon > 0$ and $\beta \geq 1$ and define $v_\epsilon = \Psi_\epsilon(\frac{u}{w \vee \epsilon}) \in D_0[\Omega] \cap L^\infty(\Omega, m)$. Then the sequence $\mu(u_n, w_n^\beta) - \mu(w_n, v_\epsilon w_n^\beta)$ converges weakly in $L^1(U_\epsilon, m)$ as $n \to +\infty$ to the function $\mu(u, w^\beta) - \mu(w, v_\epsilon w^\beta)$.*

Proof. Since $v_\epsilon = |\frac{u}{w}|^{p-2} \frac{u}{w}$ a.e. in U_ϵ, we have

$$\mu(u_n, w_n^\beta) - \mu(w_n, v_\epsilon w_n^\beta) \tag{5.4}$$

$$= \beta w_n^{\beta-1}(\mu(u_n, w_n) - \mu(\frac{u}{w} w_n, w_n))$$

$$+ \beta w_n^{\beta-1}(\mu(\frac{u}{w} w_n, w_n) - |\frac{u}{w}|^{p-2} \frac{u}{w} \mu(w_n, w_n)) - w_n^{\beta}\mu(w_n, v_\epsilon)$$

$=: A_n + B_n + C_n$ a.e. in U_ϵ. In a similar way, we obtain

$$\mu(u, w^\beta) - \mu(w, v_\epsilon w^\beta) \tag{5.5}$$

$$= \beta w^{\beta-1}(\mu(u, w) - |\frac{u}{w}|^{p-2} \frac{u}{w} \mu(w, w)) - w^\beta \mu(w, v_\epsilon)$$

$=: A + B + C$ a.e. in U_ϵ. Concerning $A_n - A$, we have from the results in Section 5 that

$$\lim_{n\to+\infty}(\mu(u_n, w_n) - \mu(\frac{u}{w} w_n, w_n)) = 0$$

in $L^1(\Omega, m)$. Then, since the sequence w_n is bounded in $L^\infty(\Omega, m)$ and converges in $L^p(\Omega, m)$, A_n converges to A weakly in $L^1(\Omega, m)$. Concerning $B_n - B$, we have that B_n converges to B a.e. (see Theorem A.1). From (2.6), the sequence B_n is also uniformly integrable; then B_n converges to B in $L^1(\Omega, m)$. Concerning $C_n - C$, we have that C_n converges to C a.e. (see Theorem A.1) and the sequence C_n is uniformly integrable; then C_n converges to C in $L^1(\Omega, m)$ and the result follows.

Lemma 5.2. *Let* $\epsilon > 0$ *and* $\beta \geq 1$ *and define* $v_\epsilon = \Psi_\epsilon(\frac{u}{w \vee \epsilon}) \in D_0[\Omega] \cap L^\infty(\Omega, m)$. *Then for every* $\varphi \in D_0[\Omega] \cap L^\infty(\Omega, m)$, *we have*

$$\int_\Omega \mu(u_n, w_n^\beta \varphi) m(dx) - \int_\Omega \mu(w_n, v_\epsilon w_n^\beta \varphi) m(dx)$$

$$= \int_\Omega \mu(u, w^\beta \varphi) m(dx) - \int_\Omega \mu(w, v_\epsilon w^\beta \varphi) m(dx) + R_n^\epsilon,$$

where $\lim_{\epsilon\to 0} \limsup_{n\to+\infty} R_n^\epsilon = 0$.

Proof. For every $\epsilon > 0$, we have

$$\int_\Omega \mu(u_n, w_n^\beta \varphi) m(dx) - \int_\Omega \mu(w_n, v_\epsilon w_n^\beta \varphi) m(dx) = A_n^\epsilon + B_n^\epsilon + C_n^\epsilon,$$

where

$$A_n^\epsilon = \int_{U_\epsilon} \varphi \mu(u_n, w_n^\beta)(dx) - \int_{U_\epsilon} \varphi \mu(w_n, v_\epsilon w_n^\beta)(dx);$$

$$B_n^\epsilon = \int_{V_\epsilon \cup W_\epsilon} \varphi \mu(u_n, w_n^\beta)(dx) - \int_{V_\epsilon \cup W_\epsilon} \varphi \mu(w_n, v_\epsilon w_n^\beta)(dx);$$

$$C_n^\epsilon = \int_\Omega w_n^\beta \mu(u_n, \varphi)(dx) - \int_\Omega v_\epsilon w_n^\beta \mu(w_n, \varphi)(dx).$$

In a similar way, we define A^ϵ, B^ϵ, C^ϵ by replacing u_n and w_n by u and w, so

$$\int_\Omega \mu(u, w^\beta \varphi) m(dx) - \int_\Omega \mu(w, v_\epsilon w^\beta \varphi) m(dx) = A^\epsilon + B^\epsilon + C^\epsilon.$$

By Lemma 5.1, we have

$$\lim_{n \to +\infty} A_n^\epsilon = A^\epsilon, \tag{5.6}$$

for every $\epsilon > 0$. We also have

$$\lim_{n \to +\infty} C_n^\epsilon = C^\epsilon. \tag{5.7}$$

In fact (see Remark A.2), $\mu(u_n, \varphi) \to \mu(u, \varphi)$, $\mu(w_n, \varphi) \to \mu(w, \varphi)$ in $L^1(\Omega, m)$, w_n^β is bounded in Ω and converges to w^β a.e. in Ω, w^β and v^ϵ are bounded in Ω.

We now consider the term $B_n^\epsilon - B^\epsilon$. For every measurable set $E \subset \Omega$, we define

$$I_n^1(E) = \beta \int_E \varphi w_n^{\beta - 1} \mu(u_n, w_n) m(dx);$$

$$I_n^{\epsilon, 2}(E) = \beta \int_E \varphi v_\epsilon w_n^{\beta - 1} \mu(w_n, w_n) m(dx);$$

$$I_n^{\epsilon,3}(E) = \beta\int_E \varphi w_n^{\beta}\mu(w_n, v_\epsilon)m(dx).$$

In a similar way, we define $I^1(E)$, $I^{\epsilon,2}(E)$, $I^{\epsilon,3}(E)$ by replacing u_n and w_n by u and w. We have

$$|B_n^\epsilon - B^\epsilon| \le |I_n^1(V_\epsilon \cup W_\epsilon)| + |I^1(V_\epsilon \cup W_\epsilon)| \tag{5.8}$$

$$+ |I_n^{\epsilon,2}(V_\epsilon)| + |I^{\epsilon,2}(V_\epsilon)| + |I_n^{\epsilon,2}(W_\epsilon)|$$

$$+ |I^{\epsilon,2}(W_\epsilon)| + |I_n^{\epsilon,3}(V_\epsilon \cup W_\epsilon) - I^{\epsilon,3}(V_\epsilon \cup W_\epsilon)|.$$

Since $\beta \ge 1$, the sequence $w_n^{\beta-1}$ is bounded in $L^\infty(\Omega, m)$. Then by Propositions 4.14 and 4.15,

$$\lim_{\epsilon\to 0} \lim_{n\to+\infty}(|I_n^1(V_\epsilon \cup W_\epsilon)| + |I_n^{\epsilon,2}(V_\epsilon)|) = 0. \tag{5.9}$$

In a similar way, we prove

$$\lim_{\epsilon\to 0}(|I^1(V_\epsilon \cup W_\epsilon)| + |I^{\epsilon,2}(V_\epsilon)|) = 0. \tag{5.10}$$

We have $|u| \le \epsilon w$ q.e. in W_ϵ, so we also have $v_\epsilon \le \epsilon^{p-1}$ q.e. in W_ϵ. As $w^{\beta-1}$ is bounded in $L^\infty(\Omega, m)$, then we have $|I_n^{\epsilon,2}(W_\epsilon)| \le K\epsilon^{p-1}\int_\Omega \alpha(w_n)$ $m(dx)$ for a suitable constant K. As w_n is bounded in $D_0[\Omega]$, then we conclude that

$$\lim_{\epsilon\to 0} \lim_{n\to+\infty}|I_n^{\epsilon,2}(W_\epsilon)| = 0. \tag{5.11}$$

In a similar way, we prove

$$\lim_{\epsilon\to 0}|I^{\epsilon,2}(W_\epsilon)| = 0. \tag{5.12}$$

We observe that $\mu(w_n, v_\epsilon) \to \mu(w, v_\epsilon)$ in $L^1(\Omega, m)$ (see Remark A.2), and $w_n \to w$ a.e. in Ω and is bounded in $L^\infty(\Omega, m)$. Then

$$\lim_{n\to+\infty} I_n^{\epsilon,3}(V_\epsilon \cup W_\epsilon) = I^{\epsilon,3}(V_\epsilon \cup W_\epsilon). \tag{5.13}$$

From (5.8)-(5.13), we have

$$\lim_{\epsilon\to 0} \lim_{n\to+\infty} |B_n^\epsilon - B^\epsilon| = 0. \tag{5.14}$$

We recall that $R_n^\epsilon = A_n^\epsilon - A^\epsilon + B_n^\epsilon - B^\epsilon + C_n^\epsilon - C^\epsilon$, then the result follows from (5.6), (5.7), and (5.14).

Lemma 5.3. *Let* $\zeta_n \in \mathcal{M}_0^p(\Omega)$. *Let* $\epsilon > 0$ *and* $\beta \geq (p-1)\vee 1$ *and define* $u_n^\epsilon = \dfrac{uw_n}{w \vee \epsilon}$. *Then*

$$\int_{U_\epsilon} |u_n|^{p-2} u_n w_n^\beta \varphi \zeta_n(dx) - \int_{U_\epsilon} |u_n^\epsilon|^{p-2} u_n^\epsilon w_n^\beta \varphi \zeta_n(dx)$$

converges to 0 *as* $n \to +\infty$, *for every* $\varphi \in D_0[\Omega] \cap L^\infty(\Omega, m)$.

Proof. Let $\varphi \in D_0[\Omega] \cap L^\infty(\Omega, m)$ and $r_n^\epsilon = u_n - u_n^\epsilon$. We recall that the sequences u_n and u_n^ϵ are bounded in $L^\infty(\Omega, m)$, then there exists a constant C such that $\| u_n|^{p-2} u_n \varphi - |u_n^\epsilon|^{p-2} u_n^\epsilon \varphi| \leq C |r_n^\epsilon|^{p-1}$. Since w_n is bounded in $L^\infty(\Omega, m)$, there exists a constant K such that $w_n^\beta \leq K w_n$. Then

$$\left| \int_{U_\epsilon} |u_n|^{p-2} u_n w_n^\beta \varphi \zeta_n(dx) - \int_{U_\epsilon} |u_n^\epsilon|^{p-2} u_n^\epsilon w_n^\beta \varphi \zeta_n(dx) \right|$$

$$\leq CK \int_{U_\epsilon} |r_n^\epsilon|^{p-1} w_n \zeta_n(dx) \leq CK \left(\int_{U_\epsilon} |r_n^\epsilon|^p \zeta_n(dx) \right)^{\frac{1}{q}} \left(\int_{U_\epsilon} w_n^p \zeta_n(dx) \right)^{\frac{1}{p}}.$$

The result follows from Proposition 4.13.

Lemma 5.4. *Let* $\zeta_n \in \mathcal{M}_0^p(\Omega)$. *Let* $\epsilon > 0$ *and* $\beta \geq (p-1)\vee 1$ *and define* $v_\epsilon = \Psi(\dfrac{u}{w \vee \epsilon}) \in D_0[\Omega] \cap L^\infty(\Omega, m)$, *and let*

$$E_n^\epsilon = \int_\Omega |u_n|^{p-2} u_n w_n^\beta \varphi \zeta_n(dx) - \int_\Omega v_\epsilon w_n^{\beta+p-1} \varphi \zeta_n(dx),$$

where $\varphi \in D_0[\Omega] \cap L^\infty(\Omega, m)$. *Then* $\lim_{\epsilon\to 0} \lim_{n\to+\infty} |E_n^\epsilon| = 0$.

The proof is the same as in Lemma 4.4 [9] by using the Lemma 5.3 and taking into account Propositions 4.14 and 4.15.

6. The Case $f = 1$

In this section, we will study the properties of the set $\mathcal{K}(\Omega)$ of the functions w such that

$$w \in D_0[\Omega],\ w \geq 0 \text{ q.e. in } \Omega,$$

and

$$\int_\Omega \mu(w, v) m(dx) \leq \int_\Omega v m(dx),$$

for every $v \in D_0[\Omega]$.

The results of the present section will be useful in the next section to investigate the convergence of the relaxed problems.

Let us observe that, if w_0 is the solution of the Dirichlet problem

$$\int_\Omega \mu(w_0, v) m(dx) = \int_\Omega v m(dx),$$

$w_0 \in D_0[\Omega]$, for every $v \in D_0[\Omega]$, then by Proposition 4.5, we have $0 \leq w \leq w_0$, for every $w \in \mathcal{K}(\Omega)$. As $w_0 \in L^\infty(\Omega, m)$, then the functions w in $\mathcal{K}(\Omega)$ are uniformly bounded. We will also prove that $\mathcal{K}(\Omega)$ is also weakly compact in $D_0[\Omega]$.

Given $w \in \mathcal{K}(\Omega)$, we define the Radon measure σ by

$$< \sigma, v >= \int_\Omega (v - \mu(w, v)) m(dx), \tag{6.1}$$

so $\sigma \in D^{-1}[\Omega]$ and is non-negative, then it is a non-negative Radon measure.

Our aim in this section is to prove the characterization of $\mathcal{K}(\Omega)$ as the set of the solutions of all relaxed Dirichlet problems (3.2) corresponding to $f = 1$.

Theorem 6.1. *The set* $\mathcal{K}(\Omega)$ *is compact in the weak topology of* $D_0[\Omega]$. *Moreover, a function* $w \in D_0[\Omega]$ *belongs to* $\mathcal{K}(\Omega)$, *if and only if there exists a measure* $\zeta \in \mathcal{M}_0^p(\Omega)$ *uniquely determined by w such that w is the solution of the relaxed Dirichlet problem* (3.2) *relative to the Borel measure* ζ. *The measure* ζ *is uniquely determined by* $w \in \mathcal{K}(\Omega)$. *More precisely, for every* $w \in \mathcal{K}(\Omega)$ *and for every Borel set* $B \subset \Omega$, *it results*

$$\zeta(B) = \int_B \frac{d\sigma}{w^{p-1}}, \text{ if } cap_p(B \cap \{w = 0\}, \Omega) = 0, \tag{6.2}$$

$$\zeta(B) = +\infty, \text{ if } cap_p(B \cap \{w = 0\}, \Omega) > 0,$$

where σ *is the non-negative Radon measure defined in* (6.1).

Before to prove Theorem 6.1, let us observe that from (6.2), we have

$$\sigma(B \cap \{w > 0\}) = \int_B w^{p-1} \zeta(dx),$$

for every Borel set $B \subset \Omega$.

To prove Theorem 6.1, we need some preliminaries results.

Lemma 6.2. *Let* $\zeta \in \mathcal{M}_0^p(\Omega)$ *and let* $u \in D_0[\Omega] \cap L^p(\Omega, \zeta)$. *Let* $u_n \in D_0[\Omega] \cap L^p(\Omega, \zeta)$ *be the solution of the problem*

$$\int_\Omega \mu(u_n, v) m(dx) + \int_\Omega |u_n|^{p-2} u_n v \zeta(dx)$$

$$+ n \int_\Omega |u_n|^{p-2} u_n v m(dx) = n \int_\Omega |u|^{p-2} u v m(dx),$$

for every $v \in D_0[\Omega] \cap L^p(\Omega, \zeta)$. *Then* u_n *converges to u strongly in* $D_0[\Omega]$ *and in* $L^p(\Omega, \zeta)$.

Lemma 6.3. *Let* $\zeta \in \mathcal{M}_0^p(\Omega)$ *and let* w *be the solution of the problem* (3.2) *with* $f = 1$. *Then* $\zeta(B) = \infty$, *for every Borel set* $B \subset \Omega$ *with* cap_p $(B \cap \{w = 0\}) > 0$.

The proofs are the same as in [9], since they depend on Theorem 4.2 and on the quasi-continuity of the functions in $D_0[\Omega]$, [5], but do not depend on special properties of the form.

Lemma 6.4. *Let* $\lambda, \nu \in \mathcal{M}_0^p(\Omega)$. *Assume that there is a function* w *in* $D_0[\Omega] \cap L^p(\Omega, \lambda) \cap L^p(\Omega, \nu)$ *such that*

$$\int_\Omega \mu(w, v) m(dx) + \int_\Omega |w|^{p-2} wv\lambda(dx) = \int_\Omega vm(dx), \tag{6.3}$$

$$\int_\Omega \mu(w, v) m(dx) + \int_\Omega |w|^{p-2} wv\nu(dx) = \int_\Omega vm(dx), \tag{6.4}$$

for every $v \in D_0[\Omega] \cap L^p(\Omega, \lambda) \cap L^p(\Omega, \nu)$. *Then* $\lambda = \nu$.

The proof is the same as in [9], since it depends on comparison principles, on Proposition 1.1, and on the quasi-continuity of the functions in $D_0[\Omega]$, [5], but does not depend on special properties of the form.

Proof of Theorem 6.1. At first, we prove that $\mathcal{K}(\Omega)$ is compact in the weak topology of $D_0[\Omega]$. Let w_n be a sequence in $\mathcal{K}(\Omega)$. Since $\mathcal{K}(\Omega)$ is bounded in $D_0[\Omega]$, we may assume that w_n converges weakly in $D_0[\Omega]$ to a function w. We have to prove that $w \in \mathcal{K}(\Omega)$.

Consider $< \sigma_n, v >= \int_\Omega vm(dx) - \int_\Omega \mu(w_n, v)m$, $v \in D_0[\Omega] \cap C_0(\Omega)$; σ_n is a bounded sequence of positive elements in $D^{-1}[\Omega]$. Then σ_n is also a bounded sequence of Radon measures, i.e., $\sigma_n(K)$ is bounded for every compact set $K \subset \Omega$. By Remark A.2, we have $\int_\Omega \mu(w_n, v)m(dx) \to \int_\Omega \mu(w, v)m(dx)$, for every $v \in D_0[\Omega]$. Then

$$\int_\Omega \mu(w, v)m(dx) \leq \int_\Omega vm(dx),$$

for every $v \in D_0[\Omega]$. From the comparison principles, we have $w \geq 0$ q.e. in Ω. Then $w \in \mathcal{K}(\Omega)$.

As second step, we assume that $\zeta \in \mathcal{M}_0^p(\Omega)$ and that w is a solution of (3.2) with $f = 1$, and we prove that $w \in \mathcal{K}(\Omega)$.

From the comparison principles, we have $w \geq 0$, then for every $v \geq 0$, we have $\int_\Omega | w |^{p-2} wv\zeta(dx) \geq 0$, so $\int_\Omega \mu(w, v)m(dx) \leq \int_\Omega vm(dx)$. Then $w \in \mathcal{K}(\Omega)$.

As third step, we assume $w \in \mathcal{K}(\Omega)$ and we prove that there exists $\zeta \in \mathcal{M}_0^p(\Omega)$ such that w is a solution of (3.2) relative to ζ and $f = 1$. The proof is analogous to the one given in [9], since it is founded only on the properties of the measure ζ and on the quasi-continuity of w.

Lemma 6.5. *Let* $\zeta \in \mathcal{M}_0^p(\Omega)$, *let* w *be the solution of* (3.2) *relative to* ζ *and* $f = 1$, *and let* $\beta \geq 1$. *Then, the set* $\{w^\beta\varphi | \varphi \in D_0[\Omega] \cap C_0(\Omega)\}$ *is dense in* $D_0[\Omega] \cap L^p(\Omega, \zeta)$.

Proof. We have $w \in D_0[\Omega] \cap L^p(\Omega, \zeta) \cap L^\infty(\Omega, m)$ and $\beta \geq 1$, then the function $w^\beta\varphi$ is in $D_0[\Omega] \cap L^p(\Omega, \zeta) \cap L^\infty(\Omega, m)$ for every $\varphi \in D_0[\Omega] \cap C_0(\Omega)$.

To prove the result, we have to find for every function $u \in D_0[\Omega] \cap L^p(\Omega, \zeta)$, a sequence $\varphi_n \in D_0[\Omega] \cap C_0(\Omega)$ such that $w^\beta\varphi_n$ converges to u both in $D_0[\Omega]$ and in $L^p(\Omega, \zeta)$. By a separation of the positive and negative part and by an approximation by truncation, we may assume $u \in L^\infty(\Omega, m)$ and $u \geq 0$ q.e. in Ω.

Let u_n be defined as in Lemma 6.2. By the comparison principles, we have $0 \leq u_n \leq Cw$ q.e. in Ω, where $C^{p-1} = n\| u \|^{p-1}_{L^\infty(\Omega)}$. From Lemma 6.2, u_n converges to u both in $D_0[\Omega]$ and in $L^p(\Omega, \zeta)$. As consequence, we may assume without loss of generality that there exists a constant C such that $0 \leq u \leq Cw$ q.e. in Ω. We observe that $\{(u - C\epsilon)^+ > 0\} \subset \{w > \epsilon\}$. Let $u_\epsilon = (u - C\epsilon)^+$ for arbitrary $\epsilon > 0$. We have $\frac{u_\epsilon}{w^\beta} = \frac{u_\epsilon}{(w \wedge \epsilon)^\beta}$.

We recall that $u_\epsilon \in D_0[\Omega] \cap L^\infty(\Omega, m)$, then $\frac{u_\epsilon}{w^\beta} \in D_0[\Omega] \cap L^\infty(\Omega, m)$.

There exists a sequence $\varphi_{n,\epsilon} \in D_0[\Omega] \cap C_0(\Omega)$ bounded in $L^\infty(\Omega, m)$, which converges to $z_\epsilon = \frac{u_\epsilon}{w^\beta}$ in $D_0(\Omega)$, then q.e. in Ω, then also ζ- a.e. in Ω.

We recall that $w \in D_0[\Omega] \cap L^\infty(\Omega, m)$ and $\beta \geq 1$, then $w^\beta \varphi_{n,\epsilon}$ converges to $w^\beta z_\epsilon = u_\epsilon$ in $D_0[\Omega]$.

We want to prove that, it also converges in $L^p(\Omega, \zeta)$. We have that $w^\beta \varphi_{n,\epsilon}$ is bounded in $L^\infty(\Omega, m) \cap L^p(\Omega, \zeta)$, then is bounded in $L^\infty(\Omega, \zeta)$. Moreover, it converges ζ- a.e. to $w^\beta z_\epsilon = u_\epsilon$. Then, it converges strongly in $L^p(\Omega, \zeta)$ (use the dominated convergence theorem).

As u_ϵ converges to u as $\epsilon \to 0$ both in $D_0[\Omega]$ and in $L^p(\Omega, \zeta)$, the result follows.

7. The γ^μ- convergence

7.1. Definition of the γ^μ- convergence

In this section, we introduce the notion of γ^μ- convergence in $\mathcal{M}_0^p(\Omega)$. This enables us to conclude about the object of the paper.

Definition 7.1. Let ζ_n be a sequence in $\mathcal{M}_0^p(\Omega)$ and let $\zeta \in \mathcal{M}_0^p(\Omega)$. We say that ζ_n γ^μ-converges to ζ, if for every $f \in D^{-1}[\Omega]$, the solutions u_n of the problem (3.2$_n$) relative to f and ζ_n converge weakly in $D_0[\Omega]$ as $n \to +\infty$ to the solution u of the problem (3.2) relative to f and ζ.

Remark 7.2. The solution of the problem (3.2$_n$) depends continuously on f uniformly with respect to ζ_n (Theorem 4.3). Then a sequence ζ_n γ^μ-converges to ζ, if the solution of the problem (3.2$_n$) relative to f and ζ_n weakly converges in $D_0[\Omega]$ to the solution of the problem (3.2), for every f in a dense subset of $D^{-1}(\Omega)$ as $L^\infty(\Omega)$.

Let ζ_n be a sequence in $\mathcal{M}_0^p(\Omega)$ and let w_n be the solution of the problem (3.2$_n$) relative to $f = 1$, and let w be the solution of the problem (3.2) relative to $f = 1$.

Theorem 7.3. *Let* $\zeta_n, \zeta \in \mathcal{M}_0^p(\Omega)$, *and let* $w_n(w)$ *be solution of* (3.2$_n$), (3.2) *relative to* $f = 1$ *and* $\zeta_n(\zeta)$. *The following conditions are equivalent*:

(a) w_n *weakly converges to* w *in* $D_0[\Omega]$.

(b) ζ_n γ^μ-*converges to* ζ.

Proof. The implication (b) $\Rightarrow$ (a) is direct consequence of the definition of γ^μ-convergence taking $f = 1$.

Assume that (a) holds. Given $f \in L^\infty(\Omega)$, let u_n be the solution of the problem (3.2$_n$). From Theorem 4.1, we have that u_n is bounded in $D_0[\Omega]$, then we may assume that u_n weakly converges to some function $u \in D_0[\Omega]$.

We have to prove that u is a solution of (3.2).

By the comparison principles, we have $|u_n| \le Cw_n$ q.e. in Ω, where $C = \| f \|_{L^\infty(\Omega)}^{\frac{1}{p-1}}$. As $n \to +\infty$, we have $|u| \le Cw$ q.e. in Ω.

For $\epsilon > 0,$ let Ψ_ϵ be the locally Lipschitz function defined in Section 5 and define $v_\epsilon = \Psi_\epsilon(\frac{u}{w \vee \epsilon})$. We have $v_\epsilon \in D_0[\Omega] \cap L^\infty(\Omega, m)$. Let $\beta \geq (p-1) \vee 1$ and let $\varphi \in D_0[\Omega] \cap C_0(\Omega)$. We recall that $w_n \in D_0[\Omega] \cap L^\infty(\Omega)$, so we can take $v = w_n^\beta \varphi$ as test function in (3.2n) and $v = v_\epsilon w_n^\beta \varphi$ as test function in (3.2n) relative to $f = 1$. We obtain

$$\int_\Omega \mu(u_n, w_n^\beta \varphi) m(dx) - \int_\Omega \mu(w_n, v_\epsilon w_n^\beta \varphi) m(dx) \tag{7.1}$$

$$+ \int_\Omega |u_n|^{p-2} u_n w_n^\beta \varphi \zeta_n(dx) - \int_\Omega |w_n|^{p-2} w_n v_\epsilon w_n^\beta \varphi \zeta_n(dx)$$

$$= \int_\Omega f w_n^\beta \varphi m(dx) - \int_\Omega v_\epsilon w_n^\beta \varphi m(dx).$$

From Lemmas 5.2 and 5.4, we obtain

$$\int_\Omega \mu(u_n, w_n^\beta \varphi) m(dx) - \int_\Omega \mu(w_n, v_\epsilon w_n^\beta \varphi) m(dx) \tag{7.2}$$

$$+ \int_\Omega |u_n|^{p-2} u_n w_n^\beta \varphi \zeta_n(dx) - \int_\Omega w_n^{\beta+p-1} \varphi \zeta_n(dx)$$

$$= \int_\Omega \mu(u, w^\beta \varphi) m(dx) - \int_\Omega \mu(w, v_\epsilon w^\beta \varphi) m(dx) + R_n^\epsilon,$$

with $\lim_{\epsilon \to 0} \limsup_{n \to +\infty} |R_n^\epsilon| = 0.$

As w_n is bounded in $D_0[\Omega]$, then it converges strongly to w in $L^p(\Omega, m)$. As consequence for every $\epsilon > 0$, we have

$$\lim_{n \to +\infty} \left(\int_\Omega f w_n^\beta \varphi m(dx) - \int_\Omega v_\epsilon w_n^\beta \varphi m(dx) \right)$$

$$= \int_\Omega f w^\beta \varphi m(dx) - \int_\Omega v_\epsilon w^\beta \varphi m(dx).$$

The above limit gives

$$\int_{\Omega}\mu(u,\ w^{\beta}\varphi)m(dx)-\int_{\Omega}\mu(w,\ v_{\epsilon}w^{\beta}\varphi)m(dx)$$

$$=\int_{\Omega}fw^{\beta}\varphi m(dx)-\int_{\Omega}v_{\epsilon}w^{\beta}\varphi m(dx)+R^{\epsilon},$$

where $\lim_{\epsilon\to 0}|R^{\epsilon}|=0$.

Defining $<\sigma, v>=\int_{\Omega}[v-\mu(w, v)]m(dx)$, we have that $\sigma\in D^{-1}[\Omega]$ defines a non-negative Radon measure. From the last inequality, we have

$$\int_{\Omega}\mu(u,\ w^{\beta}\varphi)m(dx)+\int_{\Omega}v_{\epsilon}w^{\beta}\varphi\sigma(dx)=\int_{\Omega}fw^{\beta}\varphi m(dx)+R^{\epsilon}. \tag{7.3}$$

We recall that $|u|\leq Cw$ q.e. in Ω, then from (5.3), we have $v_{\epsilon}\leq (C\vee\epsilon)^{(p-1)}$ q.e. in Ω. Recalling the definition of Ψ_{ϵ}, we obtain the convergence q.e. in Ω of $v_{\epsilon}w^{\beta}$ to $|u|^{p-2}uw^{\beta-p+1}$. As $v_{\epsilon}w^{\beta}$ is bounded in $L^{\infty}(\Omega, m)$, we have $\lim_{\epsilon\to 0}\int_{\Omega}v_{\epsilon}w^{\beta}\varphi\sigma(dx)=\int_{\Omega}|u|^{p-2}uw^{\beta-p+1}\varphi\sigma(dx)$. From (7.3), we have

$$\int_{\Omega}\mu(u,\ w^{\beta}\varphi)m(dx)+\int_{\Omega}|u|^{p-2}uw^{\beta-p+1}\varphi\sigma(dx)=\int_{\Omega}fw^{\beta}\varphi m(dx).$$

From Theorem 6.1, we have

$$\int_{\Omega}|u|^{p-2}uw^{\beta-p+1}\varphi\sigma(dx)=\int_{\{w>0\}}|u|^{p-2}uw^{\beta-p+1}\varphi\sigma(dx)$$

$$=\int_{\Omega}|u|^{p-2}uw^{\beta}\varphi\zeta(dx).$$

Then from (7.3), we obtain

$$\int_{\Omega}\mu(u,\ w^{\beta}\varphi)m(dx)+\int_{\Omega}|u|^{p-2}uw^{\beta}\varphi\zeta(dx)=\int_{\Omega}fw^{\beta}\varphi m(dx).$$

Taking into account Lemma 6.5, we have that u is the solution of (3.2) relative to ζ and f. Then ζ_n γ^{μ}-converges to ζ.

Remark 7.4. The uniqueness of the γ^{μ}- limit is an easy consequence of Theorem 7.3 and Lemma 6.4.

7.2. Compactness and density results

The following result proves the compactness of $\mathcal{M}_0^p(\Omega)$ with respect to the γ^{μ}- convergence.

Theorem 7.5. *Every sequence in* $\mathcal{M}_0^p(\Omega)$ *contains a* γ^{μ}- *convergent subsequence.*

Proof. The result follows easily from Theorems 6.1 and 7.3.

The case of Dirichlet problems in perforated domains is a particular case and it is considered in the following theorem, which is a consequence of Theorem 7.5.

Theorem 7.6. *Let* Ω_n *be an arbitrary sequence of open subsets of* Ω. *Then there exists a subsequence, still denoted by* Ω_n, *and a measure* $\zeta \in \mathcal{M}_0^p(\Omega)$ *such that for every* $f \in D^{-1}[\Omega]$, *the solution* u_n *of the problem* $\int_{\Omega_n} \mu(u_n, v) m(dx) = < f, v >_{D'[\Omega_n], D_0[\Omega_n]}, u_n \in D_0[\Omega_n]$, *for every* $v \in D_0 \ [\Omega_n]$, *extended by* 0 *to* Ω, *converges weakly in* $D_0[\Omega]$ *to the solution* u *of the problem* (3.2) *relative to a suitable Borel measure* $\zeta \in \mathcal{M}_0^p(\Omega)$.

Proof. The conclusion follows easily from Theorem 7.5 and Remark 7.2.

Theorem 7.7. *Every measure* $\zeta \in \mathcal{M}_0^p(\Omega)$ *is the* γ^{μ}- *limit of a sequence* ζ_n *of Radon measures in* $\mathcal{M}_0^p(\Omega)$ *such that the sequence of solutions* w_n *of the problem* (3.2_n) *relative to* $f = 1$, *and* ζ_n *converges strongly in* $D_0[\Omega]$ *to the solution of the problem* (3.2) *relative to* $f = 1$ *and* ζ.

Proof. By (6.2), a measure $\zeta \in \mathcal{M}_0^p(\Omega)$ is a Radon measure, if the solution w of the problem (3.2) relative to $f = 1$ and ζ is such that $\inf_K w > 0$, for every compact set $K \subset \Omega$.

We denote by $w_0 \in D_0[\Omega]$, the solution of the equation $\int_\Omega \mu(w_0, v)m(dx) = \int_\Omega vm(dx)$ for every $v \in D_0[\Omega]$, then w_0 satisfies the above inequality (Corollary 4.5).

Fix $\zeta \in \mathcal{M}_0^p(\Omega)$ and denote by $w \in \mathcal{K}(\Omega)$, the solution of the problem (3.2) relative to $f = 1$ and ζ. We define $w_n = w \vee \frac{1}{n} w_0$. It is easy to see that w_n is a nonnegative subsolution of the equation defining w_0, so $w_n \in \mathcal{K}(\Omega)$. Moreover, the function w_n satisfies the inequality $\inf_K w_n > 0$, for every compact set $K \subset \Omega$ and converges strongly to w in $D_0[\Omega]$. Then the measures ζ_n associated to w_n, which are Radon measures, γ^μ converge to ζ by Theorem 7.3.

The following result deals with the convergence of the solutions and energies, when also f varies.

Theorem 7.8. *Let ζ_n be a sequence of measures in $\mathcal{M}_0^p(\Omega)$, which γ^μ-converges to the measure $\zeta \in \mathcal{M}_0^p(\Omega)$ and let f_n be a sequence in $D^{-1}[\Omega]$, which converges to $f \in D^{-1}[\Omega]$. Define u_n as the solution of the problem* (3.2_n) *relative to f_n and ζ_n, and u as the solution of the problem* (3.2) *relative to f and ζ. Then, the sequence u_n converges to u weakly in $D_0[\Omega]$ and strongly in $D^r[\Omega]$. Finally, the energies $\alpha(u_n)m + |u_n|^p \zeta_n$ converge to $\alpha(u)m + |u|^p \zeta$ weakly* in the sense of Radon measures on Ω.*

Proof. It is enough to prove that

$$\lim_{n\to+\infty}\left(\int_\Omega \phi\alpha(u_n)m(dx) + \int_\Omega \phi|u_n|^p \zeta_n(dx)\right)$$

$$= \left(\int_\Omega \phi \alpha(u) m(dx) + \int_\Omega \phi |u|^p \zeta(dx) \right).$$

For every $\phi \in D_0[\Omega] \cap C_0(\Omega)$. The proof of the above relation is the same as in [9] taking into account Theorem A.1 and Remark A.2.

7.3. Localization properties

We end the section by proving the local character of the γ^μ-convergence. The following result deals with the local solutions in an open subset U of Ω, and we do not pay any care to the boundary conditions on ∂U. In the following, we denote by $< .,. >_U$, the pairing between $D^{-1}[U]$ and $D_0[U]$.

Theorem 7.9. *Let ζ_n be a sequence of measures in $\mathcal{M}_0^p(\Omega)$, which γ^μ-converges to the measure $\zeta \in \mathcal{M}_0^p(\Omega)$. Let U be an open subset of Ω, let f_n be a sequence in $D^{-1}[U]$, which converges to $f \in D^{-1}[U]$. Define u_n as the solution of the problem*

$$\int_U \mu(u_n, v) m(dx) + \int_U |u_n|^{p-2} u_n v \zeta_n(dx) =< f_n, v >_U, \tag{7.4}$$

$u_n \in D[U] \cap L^p(U', \zeta_n)$, for every $U' \subset\subset U$, for every $v \in D_0[U] \cap L^p$ (U, ζ_n) with $\operatorname{supp}(v) \subset\subset U$, and u as the solution of the problem

$$\int_U \mu(u, v) m(dx) + \int_U |u|^{p-2} uv \zeta(dx) =< f, v >_U, \tag{7.5}$$

$u \in D[U] \cap L^p(U', \zeta)$, for every $U' \subset\subset U$, for every $v \in D_0[U] \cap L^p$ (U, ζ) with $\operatorname{supp}(v) \subset\subset U$. We have that u_n converges weakly to u in $D_{loc}(U)$, strongly in $D^r[\Omega]$, $1 < r < p$, and $\mu(u_k, v)$ converges in L^1_{loc} (U) to $\mu(u, v)$ or every $v \in D_0[U]$ with $\operatorname{supp}(v) \subset\subset U$. Finally, the energies $\alpha(u_n)m + |u_n|^p \zeta_n$ converge to $\alpha(u)m + |u|^p \zeta$ weakly in the sense of Radon measures on U.*

Proof. Fix an open set $U' \subset\subset U$ and a function $\psi \in D_0[U] \cap L^{\infty}(U, m)$ with $\psi \geq 0$ on U, $\psi = 1$ on U', $\operatorname{supp}\psi \subset U$.

We use $v = \psi u_n$ as test function in (7.4), and we obtain

$$\int_{U'} |u_n|^p \zeta_n(dx) \leq < f_n, \psi u_n >_U - \int_U \mu(u_n, \psi u_n) m(dx) \leq M,$$

for a suitable constant M. By Theorem A.1, the sequence u_n converges to u weakly in $D_{loc}[U]$ and $\alpha(u_n - u)^{\frac{1}{p}}$ converges strongly to 0 in $L^r(U, m)$, for $1 < r < p$; moreover, $\mu(u_k, v)$ converges in $L^1(U)$ to $\mu(u, v)$ for every $v \in D_0[U]$ with $\operatorname{supp}(v) \subset\subset U$. We recall that u_n is bounded in $D[U]$. Then Remark A.2 gives the second part of the result. Define $\phi(x) = \exp\left(1 - \frac{1}{\psi(x)}\right)$, if $\psi(x) > 0$ and $\phi(x) = 0$, if $\psi(x) = 0$. Then $\phi \in D_0[U] \cap L^{\infty}(U, m)$. Let us define $z_n = \phi u_n$, $z = \phi u$. The function z_n is the solution of the problem

$$\int_{\Omega} \mu(z_n, v) m(dx) + \int_{\Omega} |z_n|^{p-2} z_n v \zeta_n(dx) = < g_n, v >, \tag{7.6}$$

$z_n \in D_0[\Omega] \cap L^p(\Omega, \zeta_n)$, for every $v \in D_0[\Omega] \cap L^p(\Omega, \zeta_n)$, where

$$< g_n, v > = \int_U \mu(\phi u_n, v) m(dx) - \int_U \phi^{p-1} \mu(u_n, v) m(dx)$$

$$+ \int_U f_n \phi^{p-1} v m(dx) - \int_U v \mu(u_n, \phi^{p-1}) m(dx).$$

Define

$$< g, v > = \int_U \mu(\phi u, v) m(dx) - \int_U \phi^{p-1} \mu(u, v) m(dx)$$

$$+ \int_U f \phi^{p-1} v m(dx) - \int_U v \mu(u, \phi^{p-1}) m(dx).$$

Taking into account (2.6), we have that $< g_n, v >$ converges to $< g, v >$ uniformly with respect to $v \in D_0[\Omega]$ with $\| v \|_{D_0[\Omega]} \leq 1$. Then g_n converges to g in $D^{-1}[\Omega]$. We recall that z_n converges to z weakly in $D_0[\Omega]$, then from Theorem 7.8, z is the solution of the problem

$$\int_\Omega \mu(z, v) m(dx) + \int_\Omega |z|^{p-2} zv \zeta(m) =< g, v >, \tag{7.7}$$

$z \in D_0[\Omega] \cap L^p(\Omega, \zeta)$, for every $v \in D_0[\Omega] \cap L^p(\Omega, \zeta)$. Since $\phi = 1$ in U', we have $u = z$ in U', then $u \in L^p(U', \zeta)$. Moreover, if $v \in D_0[\Omega] \cap L^p(\Omega, \zeta)$ with supp$v \subset\subset U'$, then $< g, v >=< g, v >_\Omega =< g, v >_U$, then (7.5) follows from (7.7). The convergence of the energies follows as in Theorem 7.8.

Theorem 7.10. *Let ζ_n be a sequence of measures in $\mathcal{M}_0^p(\Omega)$, which γ^μ-converges to the measure $\zeta \in \mathcal{M}_0^p(\Omega)$. Let U be an open subset of Ω, then ζ_n γ^μ-converges to the measure ζ in U.*

Proof. Let $f \in D^{-1}[U]$ and denote by u_n, the solution of the problem (3.2) relative to f and ζ_n with Ω replaced by U. There is a subsequence, still denoted by u_n, that converges weakly in $D_0[\Omega]$ to a function $u \in D_0[\Omega]$. From Theorem 7.9, we have $u \in L^p(U', \zeta)$ for every open set $U' \subset\subset U$ and u is a solution of (7.5).

To conclude, we have to prove that $u \in L^p(U, \zeta)$. The proof is the same as in [9], taking into account that for every $v \in D_0[U]$, there exists a sequence v_n such that v_n converges strongly to u in $D_0(\Omega)$, supp$v_n \subset\subset U$, $|v_n| \leq |u|$ q.e. in U, and $uv_n \geq 0$ q.e. in U. We also recall that, if $v \in L^p(U', \zeta)$ for every open set $U' \subset\subset U$, then $v_n \in L^p(U, \zeta)$. We may also assume that v_n converges to u q.e. in U.

Corollary 7.11. *Let* $\zeta, \zeta_n \in \mathcal{M}_0^p(\Omega)$, *and* Ω_i *be a family of open subsets of* Ω, *which covers* Ω. *Then* ζ_n γ^μ*- converges to the measure* ζ *in* Ω, *if and only if* ζ_n γ^μ*- converges to the measure* ζ *in* Ω_i *for every i.*

Proof. The conclusion follows by Theorems 7.5, 7.10, and from the uniqueness of the γ^μ- limit.

8. Appendix

Theorem A.1. *Let* $u_k \in D[\Omega]$ *with* $\int_\Omega \alpha(u_k)m(dx) \leq C$, $f_k \in D^{-1}[\Omega]$, *and* ζ_k *Radon measures, be sequences such that*

$u_k \to u$ *weakly in* $D_{loc}[\Omega]$;

$f_k \to f$ *weakly in* $D_{loc}^{-1}[\Omega]$;

$\zeta_k \to \zeta$ *weakly* in the space of Radon measures.*

Finally, we assume

$$\int_\Omega \mu(u_k, v)m(dx) =< f_k, v > + \int_\Omega v\zeta_k(dx),$$

for every $v \in D_0[\Omega] \cap C_0(\Omega)$. *Then* $\alpha(u_k - u)^{\frac{1}{p}}$ *converges strongly to* 0 *in* $L^r(\Omega, m)$, $1 < r < p$.

Proof. We observe that since the sequence u_k weakly converges to u in $D_{loc}(\Omega)$, then $u \in D[\Omega]$ and the sequence $\alpha(u_k - u)^{\frac{1}{p}}$ is bounded in $L^p(\Omega, m)$. Then to prove the result, it is enough to prove that every subsequence of $\alpha(u_k - u)^{\frac{1}{p}}$ contains a subsequence, which converges to 0 a.e. in Ω. We denote

$$g_k = \mu(u_k, u_k - u) - \mu(u, u_k - u).$$

By (2.2) and (2.4), to prove the result, it is enough to prove that g_k converges to 0 a.e. in Ω. Fix a compact $K \subset \Omega$; there exists $\phi_K \in D_0[\Omega]$ with $0 \leq \phi_K \leq 1$, $\phi_K = 1$ on K and $\alpha(\phi_K) \in L^\infty(\Omega, m)$. Let ψ_δ denote the truncation by δ, i.e.,

$$\psi(y) = y \text{ for } |y| \leq \delta; \; \psi(y) = \delta sign(y) \text{ for } |y| \geq \delta.$$

We now use as test function $v = \phi_K \psi_\delta(u_k - u) \in D_0[\Omega]$ (here, we use the truncation rule and the Leibniz inequality for the form). Then,

$$\int_\Omega \mu(u_k, \phi_K \psi_\delta(u_k - u))m(dx)$$

$$= \int_\Omega \phi_K \mu(u_k, \psi_\delta(u_k - u))m(dx) + \int_\Omega \psi_\delta(u_k - u)\mu(u_k, \phi_K)m(dx)$$

$$= \langle f_k, \phi_K \psi_\delta(u_k - u)\rangle + \int_\Omega \phi_K \psi_\delta(u_k - u)\zeta_k(dx).$$

We have that $\psi_\delta(u_k - u)$ converges to 0 strongly in $L^p(\Omega, m)$, and then weakly to 0 in $D_0[\Omega]$. Then

$$\lim_{k \to +\infty} \int_\Omega \psi_\delta(u_k - u)\mu(u_k, \phi_K)m(dx) = 0,$$

and

$$\lim_{k \to +\infty} \langle f_k, \phi_K \psi_\delta(u_k - u)\rangle = 0.$$

Moreover, ζ_k is bounded in $D'[\Omega]$, and

$$\left|\int_\Omega \phi_K \psi_\delta(u_k - u)\zeta_k(dx)\right| \leq C_K \|\psi_\delta(u_k - u)\|_{L^\infty} \leq 2C_K\delta.$$

We have so proved that, for δ fixed

$$\lim_{k \to +\infty}\left|\int_\Omega \phi_K \mu(u_k, \psi_\delta(u_k - u))m(dx)\right| \leq 2C_K\delta.$$

Since by the same methods, we have

$$\lim_{k\to+\infty}\int_\Omega \phi_K\mu(u,\,\psi_\delta(u_k-u))m(dx)=0,$$

we obtain

$$\lim_{k\to+\infty}\left|\int_\Omega \phi_K(\mu(u_k,\,\psi_\delta(u_k-u))-\mu(u,\,\psi_\delta(u_k-u)))m(dx)\right|\le 2C_K\delta.$$

Denote

$$e_k=(\mu(u_k,\,\psi_\delta(u_k-u))-\mu(u,\,\psi_\delta(u_k-u))).$$

Splitting K into the two sets $S_k^\delta=\{x\in K;\,|u_k(x)-u(x)|\le\delta\}$ and $G_k^\delta=\{x\in K;\,|u_k(x)-u(x)|>\delta\}$, and using the Hölder inequality, we obtain for $\theta<1$

$$\int_K e_k^\theta m(dx)=\int_{S_k^\delta}e_k^\theta m(dx)+\int_{G_k^\delta}e_k^\theta m(dx)$$

$$\le\left(\int_{S_k^\delta}e_k m(dx)\right)^\theta m(S_k^\delta)^{1-\theta}+\left(\int_{G_k^\delta}e_k m(dx)\right)^\theta m(G_k^\delta)^{1-\theta}.$$

Since $m(G_k^\delta)$ tends to 0 as $k\to 0$, then e_k is bounded in $L^1(\Omega,\,m)$ and

$$\limsup_{k\to+\infty}\int_K e_k^\theta m(dx)\le(2C_K)^\theta m(\Omega)^{1-\theta}\delta^\theta.$$

As $\delta>0$ is an arbitrary, the above relation implies that e_k^θ converges strongly to 0 in $L^1(K,\,m)$, when $k\to+\infty$. Then, since K is an arbitrary compact subset of Ω, at least after extraction of a subsequence, we have $e_k(x)\to 0$ a.e. in Ω, and this concludes the proof.

Remark A.2. From the previous result, we obtain that the sequence $\mu(u_k,\,v)$ converges to $\mu(u,\,v)$ pointwise a.e. in Ω for every fixed $v\in D_0[\Omega]$. The assumptions (2.3), (2.5) imply that the functions $\mu(u_k,\,v)$ are uniformly integrable. Then, the sequence $\mu(u_k,\,v)$ converges to $\mu(u,\,v)$ strongly in $L^1(\Omega,\,m)$ for every fixed $v\in D_0[\Omega]$.

References

[1] M. Biroli and N. Tchou, Asymptotic behavior of relaxed Dirichlet problems involving a Dirichlet-Poincaré form, J. Anal. Appl. 16 (1997), 281-309.

[2] M. Biroli, C. Picard and N. Tchou, Homogenization of the p-Laplacian associated with the Heisenberg group, Mem. di Mat., Rend. Acc. Naz. Sc. Detta dei XL 22 (1998), 23-42.

[3] M. Biroli and N. Tchou, Relaxed Dirichlet problem for the subelliptic p-Laplacian, Ann. Mat. Pura Appl. (IV), CLXXIX (2001), 39-64.

[4] M. Biroli, C. Picard and N. Tchou, Asymptotic behavior of some nonlinear subelliptic Dirichlet problems, Mem. di Mat., Rend. Acc. Naz. Sc. Detta dei XL 26 (2002), 235-252.

[5] M. Biroli and P. Vernole, Strongly local nonlinear Dirichlet functionals and forms, Adv. Math. Sci. Appl. 15 (2005), 655-682.

[6] M. Biroli and P. Vernole, A priori estimates for p-Laplacian associated with nonlinear Dirichlet forms, Nonlinear Analysis 64 (2006), 51-68.

[7] L. Boccardo and F. Murat, Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations, Nonlinear Analysis 19 (1992), 581-597.

[8] F. Dal Fabbro and S. Marchi, Γ-convergence of strongly local Dirichlet functionals, Riv. Mat. Univ. Parma, (in print).

[9] G. Dal Maso and F. Murat, Asymptotic behaviour and correctors for Dirichlet problems in perforated domains with homogeneous monotone operators, Ann. Sc. Norm. Sup. Pisa 24 (1979), 239-290.

[10] J. Maly and U. Mosco, Remarks on measure-valued Lagrangians on homogeneous spaces, Ricerche Mat. 48 (Supp.) (1999), 217-231.

■